

On dimensions of $2+1$ D abelian topological systems on Non-orientable surfaces

Ippo Orii

Kavli Institute for the Physics and Mathematics of the Universe

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What is TQFT

- A **Topological Quantum Field Theory (TQFT)** is a quantum field theory defined mathematically by **Atiyah's axioms**.
- It is **independent of the spacetime metric**.
- In this talk, "TQFT" refers to $(2 + 1)$ -dimensional bosonic TQFTs **without spin structure**.

However,

TQFT is well-established only in oriented cases.

What is TQFT

Anyons: particle-like excitations satisfying fusion rules.

→ They may exhibit spin statistics distinct from bosons or fermions.

Fusion rule:

$$a \times b = \sum_c N_{ab}^c c, \quad a, b, c \in \{\text{Anyons}\}, \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}$$

In particular, if the fusion outcome is unique:

$$a \times b = c,$$

then the set of anyons forms an abelian group. (We denote this set as $\mathcal{A}$)

- Oriented case: Anyon data $\rightarrow$ TQFT
- Unoriented case: Anyon data + anti-unitary $\mathbb{Z}_2$ action $\rightarrow$ TQFT?

What is TQFT

Anyon: simple object of Modular Tensor Category (MTC)

Fusion rule:

$$a \otimes b = \bigoplus_{c \in \mathcal{A}} N_{ab}^c c \quad a, b, c \in \mathcal{A}, \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}$$

In particular, if the fusion outcome is unique, i.e., $a \otimes b$ is also simple:

$$a \otimes b = c,$$

then the set of anyons forms an abelian group. (We denote this set as $\mathcal{A}$)

- Oriented case: $\text{MTC} \rightarrow \text{TQFT}$
- Unoriented case: $\text{MTC} + \text{anti-unitary } \mathbb{Z}_2 \text{ action} \rightarrow \text{TQFT?}$

What is TQFT

Notation: $\mathcal{A}$ forms an abelian group under fusion:

$$a \times b = c \rightarrow a + b = c$$

$$a \times \bar{a} = 1 \rightarrow a - a = 0$$

etc...

What We Show

When encoding time-reversal symmetry into a TQFT, one may encounter **anomalies**.

A necessary condition for anomaly cancellation:

$$M_a = \sum_b S_{ab} \eta(b) \in \mathbb{Z}, \quad \forall a \in \mathcal{A},$$

which can be defined with anyons (MTC) and time-reversal data.

If the TQFT exists, this coincides with:

$$M_a = \dim(V(\mathbb{RP}^2(a))),$$

ensuring we can **consistently perform cutting and gluing** of spacetime manifolds with time-reversal symmetry.

Our Result:

$$M_a = \sum_b S_{ab} \eta(b) \in \mathbb{Z}, \quad \forall a \in \mathcal{A},$$

for all abelian anyon systems.

What We Show

Anomaly in This Talk:

The term *anomaly* is widely used when we cannot consistently encode symmetry into our theories.

Let G be a symmetry group with $g \in G$.

- **In standard QFTs:**

A field ϕ (a smooth section of a bundle) is acted upon by G via a representation.

- **In 2+1D TQFTs:**

An anyon a (a simple object of MTC) is acted upon by

$$\rho_g \in \text{Aut}(\mathcal{C}) \quad \text{where } \rho : G \rightarrow \text{Aut}(\mathcal{C}).$$

→ We can define the twisted cohomology $H_{[\rho]}^3(G, \mathcal{A})$. We say that the theory has an anomaly if

$$H_{[\rho]}^3(G, \mathcal{A}) \neq 0$$

in this talk [2, 3].

Axioms

- I: Assign $V(\Sigma)$ to a closed two-dimensional surface Σ .
- II: $V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2)$ ($\rightarrow V(\emptyset) = \mathbb{C}$)
- III: $Z(M) \in V(\partial M)$. ($\rightarrow Z(M) \in \mathbb{C}$ for M :closed.)
- IV: $V(\Sigma^*) = V(\Sigma)^*$.

$$\begin{aligned} V(\text{torus} \sqcup \text{pair of pants}) &= V(\text{torus}) \otimes V(\text{pair of pants}) \\ Z(\text{cup}) &\in V(\text{disk}) \\ V(\text{torus with red dot}) &= V(\text{torus with red loop})^* \end{aligned}$$

A $(2+1)$ D TQFT is a symmetric monoidal functor:

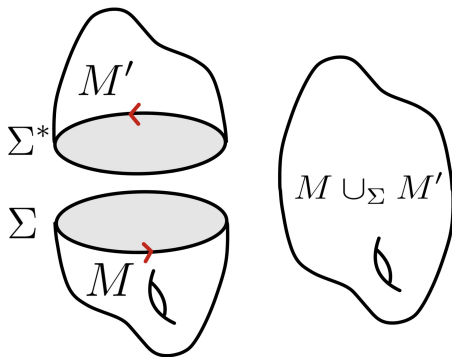
$$Z : \text{Bord}_{2,3} \longrightarrow \text{Vect}_{\mathbb{C}}.$$

Gluing:

$$|\psi\rangle := Z(M) \in V(\Sigma)$$

$$\langle\psi'| := Z(M') \in V(\Sigma)^*$$

$$\langle\psi'|\psi\rangle = Z(M \cup_{\Sigma} M') \in V(\emptyset) = \mathbb{C}$$

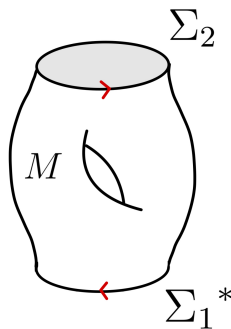


Cobordism:

$$\begin{aligned}\partial M &= \Sigma_1^* \sqcup \Sigma_2, \\ Z(M) &\in V(\Sigma_1)^* \otimes V(\Sigma_2).\end{aligned}$$

Therefore, we can write $Z(M)$ as

$$Z(M) = \sum_{a,b} U_{ab} |\psi_{\Sigma_2,a}\rangle \otimes \langle \psi_{\Sigma_1,b}|. \quad (1)$$

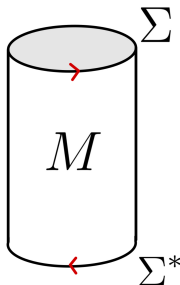


Identity cobordism:

$$M = \Sigma \times I,$$

$$\partial M = \Sigma^* \sqcup \Sigma,$$

$$Z(M) = \sum_a |\psi_{\Sigma,a}\rangle \otimes \langle \psi_{\Sigma,a}| = \text{id}_{V(\Sigma)}.$$

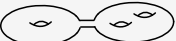


Dimension formula:

$$\begin{aligned}
 Z(\Sigma \times S^1) &= \text{Tr}[Z(\Sigma \times I)] \\
 &= \text{Tr}\left(\sum_a |\psi_{\Sigma,a}\rangle \langle \psi_{\Sigma,a}| \right) \\
 &= \dim V(\Sigma)
 \end{aligned}$$

$\forall$ orientable surface =  # ... #  or S^2 ,

$\forall$ non-orientable surface =  # ... # 

e.g.  =  # 

How to construct $V(T^2)$

To define $V(T^2)$, we consider a three-dimensional manifold N such that:

$$\partial N = T^2.$$

Then,

$$Z(N) \in V(\partial N) = V(T^2).$$

$$\rightarrow N = D^2 \times S^1 \text{ or } S^1 \times D^2:$$

$$\partial(D^2 \times S^1) = S^1 \times S^1 = T^2$$

$$\partial(S^1 \times D^2) = S^1 \times S^1 = T^2$$

How to construct $V(T^2)$

- **Constructing a basis:**

$$|a\rangle := Z(D^2(a) \times S^1) = Z(\text{disk with red dot} \times S^1) = Z(\text{red eye}) \in V(T^2),$$

then

$$V(T^2) = \text{span}_{\mathbb{C}} \{|a\rangle \mid a \in \mathcal{A}\}.$$

- **Another basis via modular S :**

$$S|a\rangle := Z(S^1 \times D^2(a)) \in V(T^2),$$

then

$$V(T^2) = \text{span}_{\mathbb{C}} \{S|a\rangle \mid a \in \mathcal{A}\}.$$

We have two complete bases of the vector space on the torus $V(T^2)$ which are related to each other via the modular S -matrix.

Analysis of Crosscap

We define the **Crosscap** state as:

$$\begin{aligned} |\mathbb{C}\mathbb{C}\rangle &= Z(\text{⊗} \times S^1) \\ &= \sum_a M_a |a\rangle \\ &= \sum_a \eta(a) S |a\rangle, \end{aligned}$$

where M_a and $\eta(a)$ are coefficients.

Then, we have

$$M_a = \langle a | \sum_b \eta(b) S |b\rangle = \sum_b S_{ab} \eta(b)$$

$$\eta(a) = \langle a | S^{-1} |\mathbb{C}\mathbb{C}\rangle$$

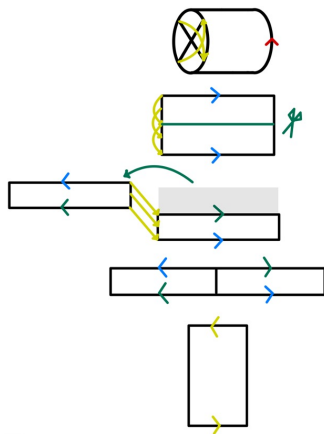


Figure: Crosscap

Analysis of Crosscap

What is M_a ?:

$$\begin{aligned}M_a &= \langle a | \mathbb{C}\mathbb{C} \rangle \\&= Z(\text{disk with red arrow} \times S^1) Z(\text{cylinder with red arrow} \times S^1) \\&= Z(\text{cylinder with two red arrows} \times S^1) \\&= Z(\text{crosscap} \times S^1) \\&= \dim V(\mathbb{RP}^2(a))\end{aligned}$$

Recall:

$$\forall \text{ non-orientable surface} = \text{crosscap} \# \cdots \# \text{crosscap}$$

Our goal is to show:

$$M_a \in \mathbb{Z} \quad \text{for any abelian systems.}$$

Abelian systems with time-reversal symmetry

- $\mathcal{A}$: a finite abelian group of anyons.
- θ : the topological spin, a function $\theta: \mathcal{A} \rightarrow U(1)$ (such that it is non-degenerate, quadratic, and homogeneous).
- c_- : the chiral central charge, an integer $c_- \in \mathbb{Z}$ satisfying the Gauss sum constraint.

Given the data above, we define the nondegenerate bilinear form by:

$$\begin{array}{ccc} B: & \mathcal{A} \times \mathcal{A} & \longrightarrow & U(1) \\ & \Psi & & \Psi \\ & (a, b) & \longmapsto & \theta(a+b)\theta(a)^{-1}\theta(b)^{-1} \end{array} . \quad (2)$$

The Gauss sum constraint is given by:

$$\frac{1}{|\mathcal{A}|^{1/2}} \sum_{a \in \mathcal{A}} \theta(a) = e^{2\pi i c_- / 8}. \quad (3)$$

Abelian systems with time-reversal symmetry

We define the time-reversal operator $T: \mathcal{A} \rightarrow \mathcal{A}$ by the condition:

$$T: \mathcal{A} \rightarrow \mathcal{A}, \quad \text{such that } \theta(Ta) \theta(a) = 1 \quad \text{for all } a \in \mathcal{A}.$$

Then, we can define Local Kramers degeneracy: We define a function

$$\eta'(a) = \begin{cases} \pm 1 & \text{if } a \in \text{Ker}(1 - T), \\ 0 & \text{otherwise,} \end{cases}$$

satisfying the following conditions:

$$\eta'(a) \eta'(b) = \eta'(a + b) \quad \text{for all } a, b \in \text{Ker}(1 - T).$$

Currently, it is understood that [7]:

$$\eta'(a) = \eta(a) = \langle a | S^{-1} | \mathbb{C}\mathbb{C} \rangle.$$

Abelian systems with time-reversal symmetry

S matrix in abelian systems:

$$S_{ab} = \frac{1}{|\mathcal{A}|^{1/2}} B(-a, b)$$

Therefore, our goal is to show

$$M_a = \frac{1}{|\mathcal{A}|^{1/2}} \sum_{b \in \text{Ker}(1-T)} B(-a, b) \eta(b) \in \mathbb{Z}$$

Main Result

Result on dimension: Consider a subset $\mathcal{M} \subset \mathcal{A}$ defined by:

$$\mathcal{M} := \{a \in \mathcal{A} \mid B(a, b) = \eta(b) \quad \text{for all } b \in \text{Ker}(1 - \mathsf{T})\}$$

Then we find:

$$M_a = \begin{cases} 2^m & \text{if } a \in \mathcal{M} \quad (\text{for some } m \in \mathbb{Z}), \\ 0 & \text{if } a \notin \mathcal{M}. \end{cases}$$

$$M_a \in \mathbb{Z} \implies \dim(V(\Sigma)) \in \mathbb{Z} \quad \text{for any non-orientable surface } \Sigma.$$

Main result

Another viewpoint:

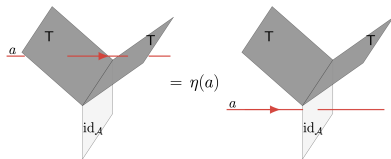
$\eta(a)$: the phase difference between the sequential local actions of T twice on an anyon $a \in \text{Ker}(1 - T)$ and the local action of their product $T^2 = \text{id}_A$.

Consider trapped anyons:

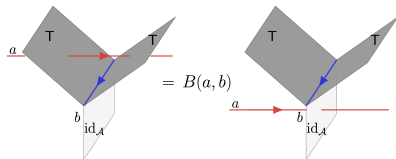
$\{a \in \mathcal{A} \mid B(a, b) = \eta(b) \ \forall b \in \text{Ker}(1 - T)\}$.

→ Reproduce the structure of $\mathcal{M}$:

$$\mathcal{M} = \{\text{Trapped anyons}\}$$



(a) Local Kramers degeneracy



(b) Trapped anyon

Generalization of the anomaly formula

Anomaly formula:

$$\theta_{\mathcal{M}} := \theta(a) = \text{const.} \in \{\pm 1\} \quad \text{for all } a \in \mathcal{M}.$$

The **anomaly** for time-reversal symmetry in 2 + 1D abelian systems is known to be classified by [5]:

$$Z(\mathbb{RP}^4) = \frac{1}{|\mathcal{A}|^{1/2}} \sum_{a \in \text{Ker}(1-T)} \theta(a) \eta(a) \in \{\pm 1\},$$

$$Z(\mathbb{CP}^2) = \frac{1}{|\mathcal{A}|^{1/2}} \sum_{a \in \mathcal{A}} \theta(a) = e^{2\pi i c_- / 8} \in \{\pm 1\}.$$

Then, let us introduce an **anomaly formula** [1, 8]:

$$Z(\mathbb{RP}^4) \cdot Z(\mathbb{CP}^2) = \theta_{\mathcal{M}}.$$

Generalization of the Anomaly Formula

Gapped Boundary:

By the axioms of TQFT, we typically assume that spaces are closed. However, what happens if the space has a boundary?

→ If the boundary itself admits a topological description, we refer to it as a **gapped boundary**.

The chiral central charge c_- provides a necessary condition for the existence of a gapped boundary:

A necessary condition for the existence of a gapped boundary is that

$$c_- \equiv 0 \pmod{8}.$$

However, it is important to note that this condition is *not sufficient*.

Generalization of the Anomaly Formula

Higher Central Charge:

Recently, the condition for the existence of a gapped boundary has been further understood via the notion of the **higher central charge**.

$$\frac{1}{|\mathcal{A}|^{1/2}} \sum_{a \in \mathcal{A}} \theta(a) = \frac{\sum_{a \in \mathcal{A}} \theta(a)}{\left| \sum_{a \in \mathcal{A}} \theta(a) \right|} \longrightarrow \xi_n := \frac{\sum_{a \in \mathcal{A}} \theta(a)^n}{\left| \sum_{a \in \mathcal{A}} \theta(a)^n \right|}$$

where $n \in \mathbb{N}$ satisfies $\gcd\left(n, \frac{2|\mathcal{A}|}{\gcd(n, 2|\mathcal{A}|)}\right) = 1$.

The invariant ξ_n provides a criterion for the existence of a gapped boundary [4]:

An Abelian TQFT admits a gapped boundary if and only if ξ_n is trivial for all n satisfying $\gcd\left(n, \frac{2|\mathcal{A}|}{\gcd(n, 2|\mathcal{A}|)}\right) = 1$.

Generalization of the Anomaly Formula

Generalization:

Define η_n as

$$\frac{1}{|\mathcal{A}|^{1/2}} \sum_{a \in \text{Ker}(1-T)} \theta(a) \eta(a) \longrightarrow \eta_n := \frac{\sum_{a \in \text{Ker}(1-T)} (\theta(a) \eta(a))^n}{\left| \sum_{a \in \text{Ker}(1-T)} (\theta(a) \eta(a))^n \right|}.$$

Compute $\eta_n \cdot \xi_n$. Then we obtain:

$$\eta_n \cdot \xi_n = \frac{\sum_{a \in \mathcal{M}^n} \theta(a)^n}{\left| \sum_{a \in \mathcal{M}^n} \theta(a)^n \right|}.$$

where

$$\mathcal{M}^n := \{a \in \mathcal{A} \mid B(a, nb) = \eta(nb) \text{ for all } b \in \text{Ker}(1 - T)\}.$$

Generalization of the anomaly formula

Another viewpoint: Recall:

$$\mathcal{M} = \{a \in \mathcal{A} \mid \dim(V(\mathbb{RP}^2(a))) \neq 0\}.$$

$\mathcal{M}^n$ can be also regarded as:

$$\mathcal{M}^n = \{a \in \mathcal{A} \mid \dim(V(\Sigma_n(na))) \neq 0\},$$

where $\Sigma_n(na)$ denotes a non-orientable surface with n crosscaps and a puncture labeled by na [1].

There are also similar structures involving topological spins, torsor structures, and other related aspects.

Reference

This talk is mainly based on [6] and my paper to appear.

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Thank you for listening.