

Proposal for a Generalization of the Anomaly Formula for Time-Reversal Symmetry in 2+1D Abelian Topological Systems



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What is TQFT

TQFT: QFTs independent of the spacetime metric.

Anyons: particle-like excitations satisfying fusion rules.

Fusion rule:

$$a \times b = \sum_c N_{ab}^c c, \quad a, b, c \in \{\text{Anyons}\}, \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}$$

If the fusion outcome is unique:

$$a \times b = c, \quad (a + b = c)$$

$\rightarrow \{\text{Anyons}\}$ forms an abelian group ($=: \mathcal{A}$.)

TQFT with time-reversal symmetry

Input: $T: \mathcal{A} \rightarrow \mathcal{A}$ anti-unitary $\mathbb{Z}_2$ action.

- Obstruction to define group action $\rightarrow \dim V(\Sigma) \notin \mathbb{Z}$.
- $Z(\mathbb{RP}^4)$ and $Z(\mathbb{CP}^2)$ classify the anomalies.
- Anomaly formula: $Z(\mathbb{RP}^4)Z(\mathbb{CP}^2) = \theta_{\mathcal{M}} \cdot [1]$

Result:

- $\dim V(\Sigma) \in \mathbb{Z} \quad \forall$ non-orientable surface Σ in abelian systems.
- Generalization of the anomaly formula.

AXIOMS of TQFT

Axiom.I: Assign $V(\Sigma)$ to closed surface Σ .

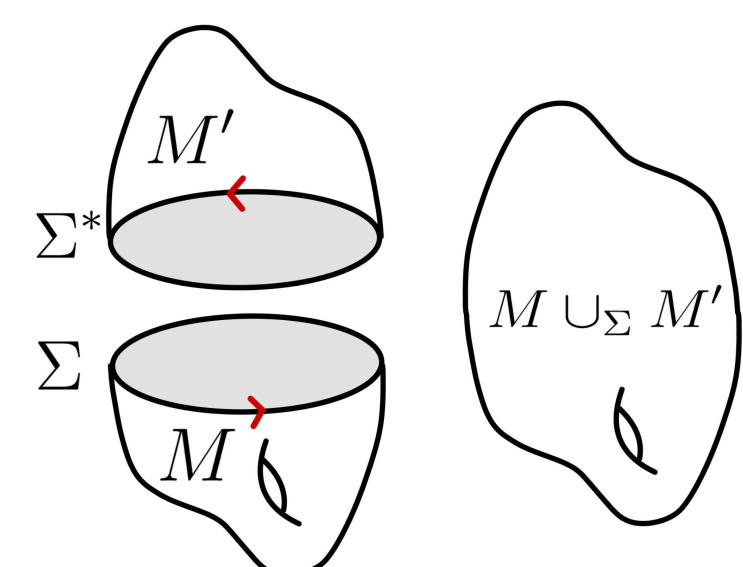
Axiom.II: $V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2) \rightarrow V(\emptyset) = \mathbb{C}$.

Axiom.III: $Z(M) \in V(\partial M) \rightarrow Z(M) \in \mathbb{C}$ for closed M .

Axiom.IV: $V(\Sigma^*) = V(\Sigma)^*$.

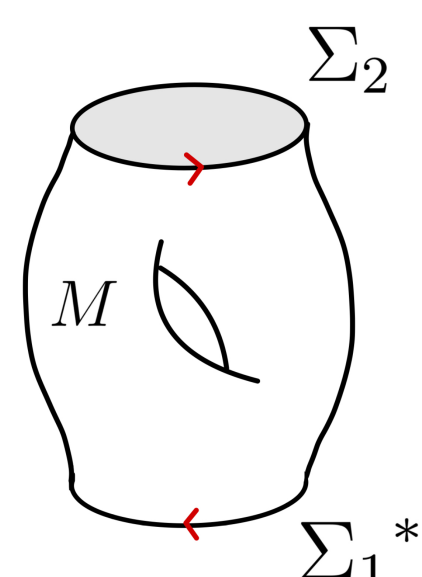
Gluing:

$$\begin{aligned} |\psi\rangle &:= Z(M) \in V(\Sigma), \\ \langle\psi'| &:= Z(M') \in V(\Sigma^*), \\ \langle\psi'|\psi\rangle &= Z(M \cup_{\Sigma} M') \in V(\emptyset) = \mathbb{C}. \end{aligned}$$



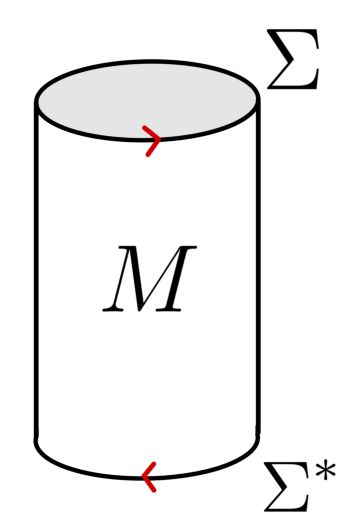
Cobordism:

$$\begin{aligned} \partial M &= \Sigma_1^* \sqcup \Sigma_2, \\ Z(M) &\in V(\Sigma_1)^* \otimes V(\Sigma_2), \\ Z(M) &= \sum_{a,b} U_{ab} |\psi_{\Sigma_2,a}\rangle \otimes \langle\psi_{\Sigma_1,b}|. \end{aligned}$$



Identity:

$$\begin{aligned} M &= \Sigma \times I, \\ \partial M &= \Sigma^* \sqcup \Sigma, \\ Z(M) &= \sum_a |\psi_{\Sigma,a}\rangle \otimes \langle\psi_{\Sigma,a}| = \text{id}_{V(\Sigma)}. \end{aligned}$$



Dimension formula: $Z(\Sigma \times S^1) = \text{Tr}[Z(\Sigma \times I)] = \dim V(\Sigma)$.

How to construct $V(T^2)$

Construction of basis:

$$|a\rangle := Z(D^2(a) \times S^1) = Z(\text{red arrow} \times S^1) = Z(\text{red oval}) \in V(T^2),$$

$$\rightarrow V(T^2) = \text{span}_{\mathbb{C}} \{|a\rangle \mid a \in \mathcal{A}\}.$$

$$S|a\rangle := Z(S^1 \times D^2(a)) \in V(T^2),$$

$$\rightarrow V(T^2) = \text{span}_{\mathbb{C}} \{S|a\rangle \mid a \in \mathcal{A}\}.$$

Why T^2 and $\mathbb{RP}^2$?:

$$\forall \text{ orientable surface} = \text{red oval} \# \cdots \# \text{red oval} \text{ or } S^2$$

$$\forall \text{ non-orientable surface} = \text{red crosscap} \# \cdots \# \text{red crosscap}$$

$$\text{e.g. } \text{red oval} \# \text{red oval} = \text{red oval} \# \text{red oval}$$

Analysis of Crosscap

Crosscap state:

$$\begin{aligned} |\mathbb{CC}\rangle &= Z(\text{red crosscap} \times S^1) \\ &= \sum_a M_a |a\rangle \\ &= \sum_a \eta(a) S|a\rangle, \\ \rightarrow M_a &= \sum_b S_{ab} \eta(b) \end{aligned} \quad \begin{aligned} M_a &= \langle a | \mathbb{CC} \rangle \\ &= Z(\text{red arrow} \times S^1) Z(\text{red crosscap} \times S^1) \\ &= Z(\text{red crosscap} \times \text{red arrow} \times S^1) \\ &= Z(\text{red crosscap} \times S^1) \\ &= \dim V(\mathbb{RP}^2(a)) \end{aligned}$$

$$M_a \in \mathbb{Z} \quad \Rightarrow \quad \dim(V(\Sigma)) \in \mathbb{Z} \quad \text{for any non-orientable surface } \Sigma.$$

Set up for Abelian systems

Defining data of abelian systems:

- $\mathcal{A}$: a finite abelian group of anyons.
- θ : the topological spin; $\mathcal{A} \rightarrow U(1)$ such that ...
- c_- : the chiral central charge $\in \mathbb{Z}$ satisfying $\frac{1}{|\mathcal{A}|^{1/2}} \sum_{a \in \mathcal{A}} \theta(a) = e^{2\pi i c_- / 8}$.

$\rightarrow$ Nondegenerate pairing: $B(a, b) = \theta(a + b) \theta(a)^{-1} \theta(b)^{-1}$,

$\rightarrow$ Modular S matrix: $S_{ab} = \frac{1}{|\mathcal{A}|^{1/2}} B(-a, b)$.

Defining data of time-reversal symmetry:

- Operator: $T: \mathcal{A} \rightarrow \mathcal{A}$, s.t $\theta(Ta) \theta(a) = 1 \quad \forall a \in \mathcal{A}$.
- Local Kramers degeneracy:

$$\eta(a) = \begin{cases} \pm 1 & \text{if } a \in \text{Ker}(1 - T), \\ 0 & \text{otherwise.} \end{cases}$$

Anomaly:

$$Z(\mathbb{RP}^4) = \frac{1}{|\mathcal{A}|^{1/2}} \sum_{a \in \text{Ker}(1-T)} \theta(a) \eta(a), \quad Z(\mathbb{CP}^2) = \frac{1}{|\mathcal{A}|^{1/2}} \sum_{a \in \mathcal{A}} \theta(a).$$

Main result

Derivation of M_a :

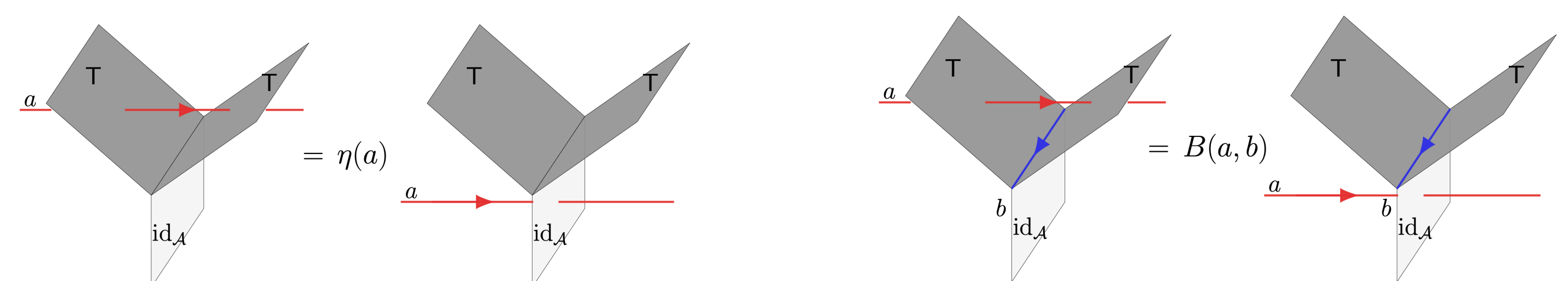
$$M_a = \frac{1}{|\mathcal{A}|^{1/2}} \sum_{b \in \text{Ker}(1-T)} B(-a, b) \eta(b) = \begin{cases} 2^m & \forall a \in \mathcal{M} \\ 0 & \forall a \in \mathcal{A} \setminus \mathcal{M} \end{cases}$$

Anomaly fomula:

$\theta(a) = \text{const.} \quad \forall a \in \mathcal{M} \quad (=:\theta_{\mathcal{M}})$ satisfying $Z(\mathbb{RP}^4)Z(\mathbb{CP}^2) = \theta_{\mathcal{M}}$.

Trapped anyon:

$\mathcal{M} = \{\text{Trapped anyons}\}: \forall a \in \text{Ker}(1 - T), \forall b \in \mathcal{M},$



Higher central charge:

$\xi_n := \sum_{a \in \mathcal{A}} \theta(a)^n / \left| \sum_{a \in \mathcal{A}} \theta(a)^n \right|$ where $n \in \mathbb{N}$ satisfies $\gcd\left(n, \frac{2|\mathcal{A}|}{\gcd(n, 2|\mathcal{A}|)}\right) = 1$.

$\rightarrow$ The existence of gapped boundaries. [2]

Generalization of Anomaly formula:

Define $\eta_n := \sum_{a \in \mathcal{A}} (\theta(a) \eta(a))^n / \left| \sum_{a \in \mathcal{A}} (\theta(a) \eta(a))^n \right|$, then

$$\eta_n \cdot \xi_n = \sum_{a \in \mathcal{M}^n} \theta(a)^n / \left| \sum_{a \in \mathcal{M}^n} \theta(a)^n \right|.$$

Here,

$$\mathcal{M}^n := \{a \in \mathcal{A} \mid \dim(V(\Sigma_n(na))) \neq 0\},$$

$\Sigma_n(na)$: a non-orientable surface with n crosscaps and a puncture na .

This poster is mainly based on arxiv: 2502.13532 and my paper to appear.

[1]:Justin Kaidi et al. arxiv: 2107.13091.

[2]:Chenjie Wang and Michael Levin. arxiv: 1610.04624.