

Reshetikhin - Turaev TQFT with time-reversal symmetry

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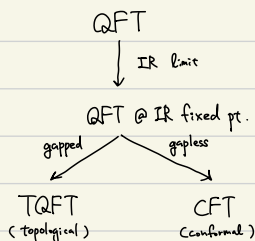
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Refs [n] n ∈ {1, ..., 17}

1. "Topological Quantum" by Steven H. Simon
2. "Quantum Invariants of Knots and 3-Manifolds" by Vladimir G. Turaev
3. "数理科学 2025年12月号"
4. "時間反転対称性位相場の理論" My master thesis I'm writing now...
5. "位相場の理論 I" by Toshinori Takaue
6. "Homeomorphisms of non-orientable two-manifolds" by W.B.R. Lickorish
7. 1403.11467
8. 1410.4540
9. 1604.06527
10. 1610.04624
11. 1611.01601
12. 1706.09464
13. 1712.09542
14. 1803.09336
15. 2502.13532
16. 2508.04990
17. 2509.07368

§1. What is TQFT?

TQFT: QFT indep of the spacetime metric
(today, TQFT refers to (2+1)D bosonic theory)



Basic concepts

Anyon: Generalized concepts of bosons / fermions

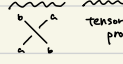
$$d_{\uparrow a}^2 = e^{2\pi i s_a} \uparrow_a \quad m \in \mathbb{Z}$$

Fusion rule: The way of the fusion of anyons

$$a \times b = \sum_c N_{ab}^c c \quad a, b, c \in \{\text{anyon}\}$$

→ Captured by Modular Tensor Category (MTC)

Properties of MTC

- $\mathcal{C}$ is a braided monoidal cat.

- Hom space is a finite diml vec sp.

$$\text{Hom}(a \otimes b, c) \cong \mathbb{C}^{\otimes N_{ab}^c}$$
- $\mathcal{C}$ is a rigid cat.

$$\text{anyon} \xleftrightarrow{\quad} \text{anti-anyon} \quad \forall a \quad \exists! \bar{a}$$
- $\mathcal{C}$ has a twist (ribbon structure)

$$d_{\uparrow a}^2 = \theta(a) \uparrow_a$$
- $\mathcal{C}$ is a semi-simple cat.

$$\forall \text{ obj} = \oplus \text{ simple obj}$$

Def

Given a MTC $\mathcal{C}$

Anyon: simple objects of $\mathcal{C}$ $a \in \text{Simp}(\mathcal{C})$

Fusion rule: decompositions of tensor products of anyons.

$$a \otimes b = \bigoplus_{c \in \text{Simp}(\mathcal{C})} N_{ab}^c N_{ab}^c$$

$$N_{ab}^c \in \mathbb{Z}_{\geq 0}$$

Recall

Addition of spin-1/2 particles in QM (representation of $\text{su}(2)$)

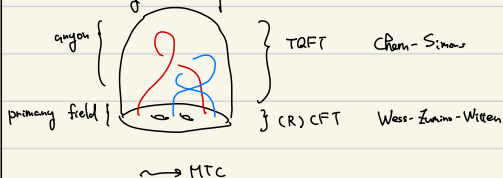
$$\mathbb{D} \otimes \mathbb{D} = \mathbb{1} \oplus \mathbb{3}$$

$\mathbb{1}, \mathbb{D}, \mathbb{3} \in \text{Irreducible rep}$

$$N_{\mathbb{1}\mathbb{1}}^{\mathbb{1}} = N_{\mathbb{1}\mathbb{1}}^{\mathbb{3}} = 1$$

RnR

Bulk-bdy correspondence btw 3D TQFT & 2D RCFT



§2. Def of TQFT

Def [2,3]

3-dim'l bosonic TQFT is a symmetric monoidal functor:

$$\mathbb{Z}: \tilde{\text{Bord}}_{2,3}^H(\mathcal{C}) \rightarrow \text{Vect}_{\mathbb{C}}$$

where $\tilde{\text{Bord}}_{2,3}^H(\mathcal{C})$: Bordism cat decorated by $\mathcal{C}$ with a tangential structure H .

Roughly speaking...

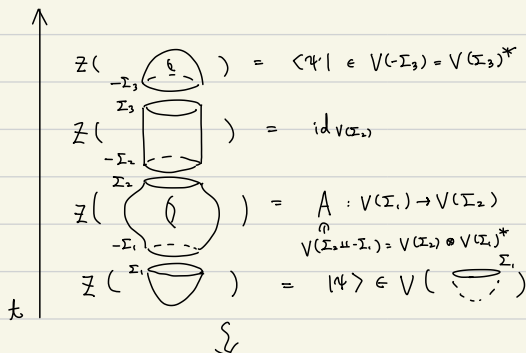
(Σ : 2-dim'l closed surface, M : 3-dim'l mfd)

$$1. V(\Sigma) := \mathbb{Z}(\Sigma) \text{ finite dim'l vec sp on } \Sigma$$

$$2. V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2) (\Rightarrow V(\emptyset) = \mathbb{C})$$

$$3. \mathbb{Z}(M) \in V(\partial M) (\Rightarrow \mathbb{Z}(M) \in \mathbb{C} \text{ wr } M: \text{closed})$$

$$4. V(-\Sigma) = V(\Sigma)^*$$



$$\mathbb{Z}(\emptyset) = \langle \psi | A | \psi \rangle \in V(\emptyset) = \mathbb{C}$$

dim formula

$$\mathbb{Z}(\Sigma \times S^1) = \text{Tr } \mathbb{Z}(\Sigma \times [0,1])$$

$$= \text{Tr } \mathbb{Z}(\text{cylinder})$$

$$= \text{Tr } \text{id}_{V(\Sigma)}$$

$$= \sum_a |\mathbb{I}_a\rangle \langle \mathbb{I}_a|$$

$$= \dim V(\Sigma)$$

Rmk

Def & construction

eg

Reshetikhin-Turaev

Turaev-Viro

Crane-Yetter

:

bulk-bdry

Rough sketch of construction

Try to construct inv for 3-mfd M

$$\langle M \rangle = \langle S^3_L \rangle \leadsto \text{diagram of } S^3 \text{ with a red and blue link}$$

$$= \sum_{a,b} \langle \text{diagram} \rangle$$

Data to evaluate Link inv

$\leadsto$ MTC does the job.!

§3. Reshetikhin-Turaev invariant. [2,3]

RT inv: $M \rightarrow \mathbb{C}$ invariant of closed oriented 3-mfd

$\rightarrow$ We can construct TQFT! (functor)

Thm (Lickorish-Wallace)

$\forall$ closed connected 3-mfd M

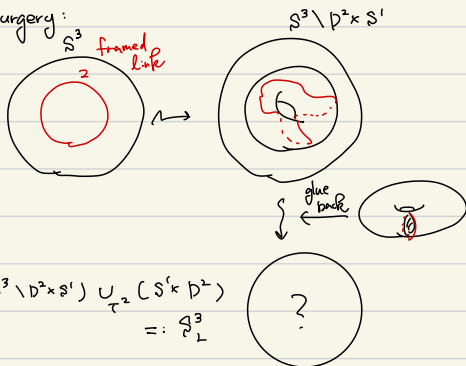
$\exists L \subset S^3$ framed Link s.t. $S^3_L \cong M$

S^3_L : 3-mfd obtained from S^3 via surgery along L .

framed link = link w/ framing number $\in \mathbb{Z}$

eg. $^2 \bigcirc ^{-3}$

Surgery:



$$(S^3 \setminus D^2 \times S^1) \cup_{\tau} (S^1 \times D^2) =: S^3_L$$

Fact (universal construction)

$\bullet \langle \rangle$: inv of 3-mfd w/ $(m), (c)$

$\Rightarrow \exists!$ Quantization function from $\langle \rangle$

here,

$$(m): \langle M \cup M' \rangle = \langle M \rangle \langle M' \rangle$$

$$(c): \langle -M \rangle = \overline{\langle M \rangle}$$

$\bullet \langle \rangle$: RT inv $\Rightarrow$ TQFT!

§4. Basics of Time reversal.

Time reversal: Anti-unitary $\mathbb{Z}_2$ grp $\sim$ MTC τ

Fact [9]

"Theory with time rev sym" = "Spacetime is nonorientable"

eg. Möbius strip, Klein bottle

Rmk [7]

Background for time rev = $w_1 \in H^1(M; \mathbb{Z}_2)$

first Stiefel-Whitney class

Obstruction for orientability.

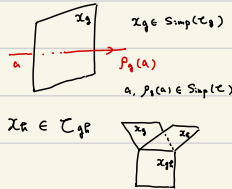
Grp action in general [8]

- G : finite grp
- $\mathcal{C}$: MTC
- $\mathcal{A}$: Subcat of invertible objects ($a \times b = c$ (unique))

$$\rho: G \rightarrow \text{Aut}(\mathcal{C})$$

Try to construct...

- $\mathcal{C}_G := \bigoplus_{g \in G} \mathcal{C}_g$
 - $\mathcal{C}_{\text{id}} = \mathcal{C}$ (anyon)
 - $\mathcal{C}_g$: branchsheet encoding grp action
 - $x_g \in \text{Simp}(\mathcal{C}_g)$
 - $a, \rho_g(a) \in \text{Simp}(\mathcal{C})$
 - $x_g \otimes x_h \in \mathcal{C}_{gh}$
 - $(x_g \otimes x_h) \otimes x_k \cong x_{ghk} \otimes e(g, h, k)$



Rmk

- $(*)$ always holds for abelian TQFT $\mathcal{C} = \mathcal{A}$ [15]
- H^3 obstruction vanishes for " [16]
- e.g. for nontrivial H^3 is known [12]

"Def"

- Time rev on $\mathcal{C}$
 - $\Leftrightarrow T: \mathcal{C} \rightarrow \mathcal{C}$ s.t. $T^2 = \text{id}_{\mathcal{C}}$, $\theta(Ta) = \overline{\theta(a)}$
 - $a \mapsto Ta$
- local Kramers degeneracy (sym fractionalisation, Frobenius-Schur indicator)
 - $\Leftrightarrow \eta: \mathcal{C} \rightarrow \{0, \pm 1\}$
 - s.t. $\eta(a) \in \{\pm 1\}$ $a = T_a$
 - $\eta(a) = 0$ $a \neq T_a$

H^3 -obstruction (postnikov class)

Obstruction $\in H^3(G, \mathcal{A})$ for constructing $\bigoplus_{g \in G} \mathcal{C}_g$

$\rightarrow$ Associativity does not hold!

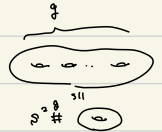
$$(x_g \otimes x_h) \otimes x_k \cong (x_g \otimes (x_h \otimes x_k)) \otimes e(g, h, k)$$

$\rightarrow$ Sym str is captured by 2-grp. [13, 14]
mixed str of 0-form 1-form sym.

Fact

$\forall$ orientable surface

$$\cong S^2 \# T^2 \# \dots \# T^2$$



$\forall$ nonorientable surface

$$\cong \mathbb{R}P^2 \# \dots \# \mathbb{R}P^2$$

$$\textcircled{\circ} \# \dots \# \textcircled{\circ}$$

e.g. Klein bottle $\textcircled{\circ}$

$$\textcircled{\otimes \otimes \otimes} \cong \textcircled{\otimes \otimes}$$

$$\textcircled{\otimes \otimes \otimes \otimes} \cong \textcircled{\otimes \otimes \otimes}$$

$\rightarrow$ Analysis of T^2 & $\mathbb{R}P^2$ is fundamental!

Fact [12]

Necessary conditions for H^3 for time rev to vanish.

$$(*) \left\{ \begin{array}{l} M_a := \mathbb{Z}(\text{RP}^2(a) \times \beta^1) \in \mathbb{Z}_{\geq 0} \quad \forall a \in \text{Simp}(\mathcal{C}) \\ \mathbb{Z}_{\text{cr}}(\mathbb{R}P^2) \in \{\pm 1\} \\ \theta_a = \text{const.} \quad \forall a \in \text{Simp}(\mathcal{C}) \text{ s.t. } M_a \neq 0 \end{array} \right.$$

§5. Analysis of Crosscap state

Construction of $V(T^2)$

We need 3-fold M s.t. $\partial M = T^2$

$$\odot \quad \mathbb{Z}(M) \in V(\partial M) = V(T^2)$$

$$\leadsto \text{solid torus } \partial(\text{torus}) = \text{circle}$$

$$\leadsto M = D^2 \times S^1, S^1 \times D^2$$

Fact

$$V(T^2) = \text{span}_{\mathbb{C}} \{ |a\rangle \mid a \in \text{Simp}(\mathbb{C}) \}$$

$$= \text{span}_{\mathbb{C}} \{ S|a\rangle \mid a \in \text{Simp}(\mathbb{C}) \}$$

$$\text{where } |a\rangle := \mathbb{Z}(D^2 \times S^1) = \mathbb{Z}(D^2 \times S^1) = \mathbb{Z}(\text{torus})$$

$$S|a\rangle := \mathbb{Z}(S^1 \times D^2)$$

Rmk

$$\dim V(T^2) = \#(\text{anyon})$$

$$\text{cf. } \dim(\text{sp of conf block on } T^2) = \#(\text{primary fld})$$

Crosscap state

$$|CC\rangle := \mathbb{Z}(\text{crosscap} \times S^1)$$

Obs

$$\text{crosscap} = \text{Möbius strip}$$

$$\odot \quad \text{crosscap} \cong \text{torus with cut}$$

$$\cong \text{torus with cut and twist}$$

$$\cong \text{torus with cut and twist}$$

Obs

$$\partial(\text{crosscap} \times S^1) = S^1 \times S^1 = T^2$$

$$\therefore |CC\rangle \in V(T^2)$$

$$\leadsto |CC\rangle \text{ can be expanded by } \{|a\rangle\}, \{S|a\rangle\}?$$

$$\leadsto |CC\rangle = \sum_a M_a |a\rangle$$

$$= \sum_a \eta(a) S|a\rangle$$

Here $M_a, \eta(a)$ are introduced as coeff.

$\rightarrow$ What are they?

About M_a

$$M_a = \langle a | CC \rangle = \left(= \sum_b \text{Sub} \eta(b) \right)$$

$$= \mathbb{Z}(D^2 \times S^1) \mathbb{Z}(\text{crosscap} \times S^1)$$

$$= \mathbb{Z}(\text{crosscap} \times D^2 \times S^1)$$

$$= \mathbb{Z}(D^2 \times S^1)$$

$$= \dim V(\mathbb{RP}^2 \times S^1) \quad \text{w/ } \mathbb{RP}^2 \times S^1 := D^2 \times S^1$$

Rmk

Why "dim"?

$\rightarrow M_a$ can be interpreted as $\dim V(\mathbb{RP}^2 \times S^1)$

only if TQFT exists.

$\rightarrow M_a \in \mathbb{Z}_{\geq 0}$ is necessary cond! (stated above)

About $\eta(a)$

$\eta(a)$: local Kramers degeneracy

$$\eta(a) = \langle a | \tilde{S}^{-1} | e \rangle$$

$$= \mathbb{Z}(\underbrace{X^3(a)}_{\text{nonorientable 3-mfld}}) \quad [11]$$

nonorientable 3-mfld $\rightarrow$ Key for the rigorous

formulation of nonorientable TQFT!

Anomaly inflow

{Anomaly in d-dim theory} $\leftrightarrow$ {SPT phase in d+1-dim theory}

In our case, we should check

$$\mathbb{Z}_{CY}(\mathbb{RP}^4) \in \{\pm 1\}$$

$$\mathbb{Z}_{CY}(\mathbb{CP}^2) \in \{\pm 1\} \quad (\Omega_4^{Spin} \cong \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2})$$

Anomaly formula [10, 15, 16]

$$\mathbb{Z}_{CY}(\mathbb{RP}^4) \cdot \mathbb{Z}_{CY}(\mathbb{CP}^2) = \theta_M$$

where $\theta_M = \theta_a$ satisfying $M_a \neq 0$

Toward the rigorous formulation

Construction of RT inv.

$$\mathbb{S}^3 \wr L \xrightarrow{\text{surj}} \mathcal{V}(\text{orientable 3-mfld}) \rightarrow \mathbb{I}\langle L \rangle$$

Surprisingly...

$$(\eta(a) \leftrightarrow) X^3 \wr L \xrightarrow{[6]} \mathcal{V}(\text{non "}) \xrightarrow{\text{why unknown...}} \text{---} ?$$

$\eta(a) \neq 0$ only for $a = Ta$

$$\rightarrow \text{We need } \begin{cases} \bullet \mathbb{Z}_2 \sim \mathbb{Z} \\ \bullet \sum_{a=Ta} \langle \phi a \rangle \end{cases} \quad ??$$

$\rightarrow$ Wait for the future works!