

Insights from the crosscap state toward a rigorous formulation of non-orientable TQFTs

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- What is TQFT
- Axioms
- How to construct $V(T^2)$
- How to encode time-reversal symmetry
- Analysis of the crosscap state
- Toward a formulation of non-orientable TQFTs

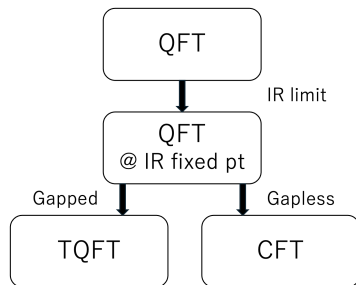
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What is TQFT

A **Topological Quantum Field Theory (TQFT)** is

- a QFT defined mathematically,
- independent of the spacetime metric,
- used to describe physics at IR fixed point.

Today, TQFT refers to $(2 + 1)$ -dimensional TQFTs without spin structures.



What is TQFT

Physical applications:

- $\{\text{Anomaly in } d\text{-dim}\} = \{\text{SPT phase in } d + 1\text{-dim}\}$
- Low-dimensional Quantum gravity
- Effective theory for fractional quantum hall effect

Example: Chern–Simons theory

$$S_{\text{CS}} = \frac{k}{4\pi} \int_{M^3} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

→ $\frac{\delta S_{\text{CS}}}{\delta g_{\mu\nu}} = 0$: independent of metric.

→ VEV of Wilson loop is computed as a knot invariant.

What is TQFT

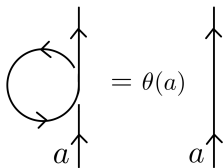
Anyons: particle-like excitations satisfying fusion rules.

→ They may exhibit spin statistics distinct from bosons or fermions.

Fusion rule:

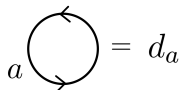
$$a \times b = \sum_c N_{ab}^c c, \quad a, b, c \in \{\text{Anyons}\} =: \mathcal{C}, \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}$$

Topological spin: $\theta : \mathcal{C} \rightarrow U(1)$ encoding spin statistics of anyons:



The diagram shows an equality between two configurations of anyon lines. On the left, a vertical line labeled 'a' at the bottom has a circular loop with two arrows (one clockwise, one counter-clockwise) encircling it. On the right, a single vertical line labeled 'a' at the bottom. The two configurations are separated by an equals sign, with the expression $\theta(a)$ placed between them.

Quantum dimension:



The diagram shows a circular loop with two arrows, one clockwise and one counter-clockwise, labeled 'a' at the bottom. This is followed by an equals sign and the symbol d_a .

What is TQFT

Bulk–Boundary correspondence: 3D TQFT $\leftrightarrow$ 2D RCFT

Anyons: primary fields in RCFT.

Fusion rule:

$$\phi_i \times \phi_j = \sum_c N_{ij}^c \phi_c, \quad i, j, c \in \{\text{primary fields}\} =: \mathcal{C}, \quad N_{ij}^c \in \mathbb{Z}_{\geq 0}$$

Topological spin: $\theta : j \mapsto e^{2\pi i h_j} \in U(1)$, conformal weight in RCFT.

Quantum dimension: VEV of Verlinde line.

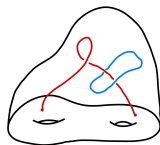


Figure: Rough picture of the correspondence

What is TQFT

Anyon: simple object of Modular Tensor Category (MTC)

Fusion rule:

$$a \otimes b = \bigoplus_{c \in \mathcal{C}} c^{\oplus N_{ab}^c} \quad a, b, c \in \mathcal{C}, \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}$$

Recall

The addition of spin-1/2 particle:

$$2 \otimes 2 = 1 \oplus 3$$

- **Orientable case:**

MTC $\rightarrow$ TQFT

- **Non-orientable case:**

MTC + anti-unitary $\mathbb{Z}_2$ action $\xrightarrow{?}$ TQFT

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Axioms

- I: Assign $V(\Sigma)$ to a closed two-dimensional surface Σ .
- II: $V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2)$ ($\rightarrow V(\emptyset) = \mathbb{C}$)
- III: $Z(M) \in V(\partial M)$. ($\rightarrow Z(M) \in \mathbb{C}$ for M :closed.)
- IV: $V(\Sigma^*) = V(\Sigma)^*$.

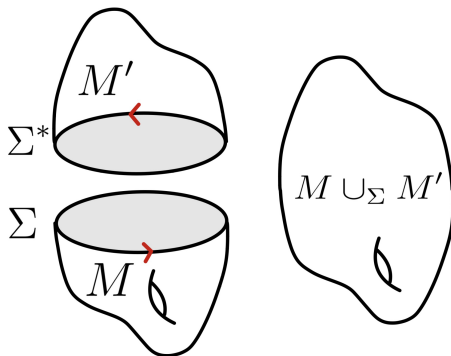
$$\begin{aligned}
 V(\text{torus} \sqcup \text{pair of pants}) &= V(\text{torus}) \otimes V(\text{pair of pants}) \\
 Z(\text{cup}) &\in V(\text{disk}) \\
 V(\text{torus with red arrow}) &= V(\text{torus with red arrow})^*
 \end{aligned}$$

Gluing:

$$|\psi\rangle := Z(M) \in V(\Sigma)$$

$$\langle\psi'| := Z(M') \in V(\Sigma)^*$$

$$\langle\psi'|\psi\rangle = Z(M \cup_{\Sigma} M') \in V(\emptyset) = \mathbb{C}$$

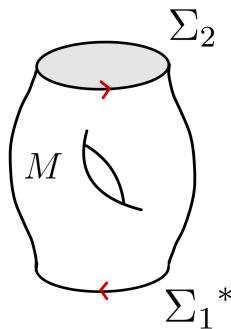


Cobordism:

$$\begin{aligned}\partial M &= \Sigma_1^* \sqcup \Sigma_2, \\ Z(M) &\in V(\Sigma_1)^* \otimes V(\Sigma_2).\end{aligned}$$

Therefore, we can write $Z(M)$ as

$$Z(M) = \sum_{a,b} U_{ab} |\psi_{\Sigma_2,a}\rangle \otimes \langle \psi_{\Sigma_1,b}|. \quad (1)$$

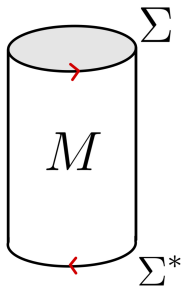


Identity cobordism:

$$M = \Sigma \times I,$$

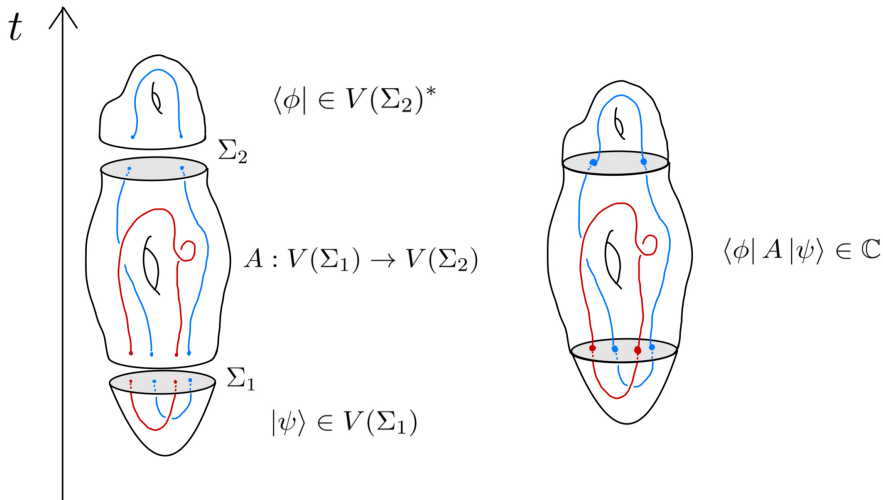
$$\partial M = \Sigma^* \sqcup \Sigma,$$

$$Z(M) = \sum_a |\psi_{\Sigma,a}\rangle \otimes \langle \psi_{\Sigma,a}| = \text{id}_{V(\Sigma)}.$$



Axioms

Summary of how to compute:



Dimension formula:

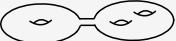
$$\begin{aligned} Z(\Sigma \times S^1) &= \text{Tr}[Z(\Sigma \times I)] \\ &= \text{Tr}\left(\sum_a |\psi_{\Sigma,a}\rangle \langle \psi_{\Sigma,a}| \right) \\ &= \dim V(\Sigma) \quad \text{for closed surface } \Sigma. \end{aligned}$$

Remark

This is equivalent as computing the ground state degeneracy.

$\forall$ orientable surface =  # ... #  or S^2 ,

$\forall$ non-orientable surface =  # ... # 

e.g.  =  # 

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How to construct $V(T^2)$

- Constructing a basis:

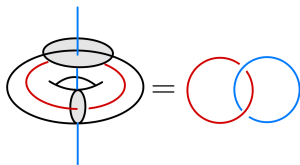
$$|a\rangle := Z(D^2(a) \times S^1) = Z(\text{disk with red dot} \times S^1) = Z(\text{torus with red loop}) \in V(T^2),$$
$$\rightarrow V(T^2) = \text{span}_{\mathbb{C}} \{|a\rangle \mid a \in \mathcal{C}\}.$$

- Another basis via modular S :

$$S|a\rangle := Z(S^1 \times D^2(a)) \in V(T^2),$$
$$\rightarrow V(T^2) = \text{span}_{\mathbb{C}} \{S|a\rangle \mid a \in \mathcal{C}\}.$$

Matrix element of S :

$$\langle a|S|b\rangle = S_{ab} = \frac{1}{\mathcal{D}} \text{ (diagram of two overlapping circles labeled } a \text{ and } b \text{ with arrows indicating a swap)}$$



Remark

$V(T^2)$ is a representation space of $SL(2, \mathbb{Z})$; $\langle S, T \rangle \curvearrowright V(T^2)$.

How to construct $V(T^2)$

Bulk–Boundary correspondence: 3D TQFT $\leftrightarrow$ 2D RCFT

$$V(\Sigma) \cong \text{Space of the conformal block } \mathcal{H}(\Sigma)$$

Example: Level k $\mathfrak{su}(2)$ affine Lie algebra:

$$\mathcal{H}(T^2) := \bigoplus_{i=0}^k \mathcal{H}_i, \quad \chi_i(\tau) := \text{Tr}_{\mathcal{H}_i} q^{L_0 - \frac{c}{24}}, \quad q = e^{2\pi i \tau}.$$

$$\begin{aligned} \chi_i(-\frac{1}{\tau}) &= \sum_j S_{ij} \chi_j(\tau), & \chi_i(\tau + 1) &= e^{2\pi i h_i} \cdot e^{-2\pi i \frac{c}{24}} \chi_i(\tau) \\ S |a\rangle &= \sum_b S_{ab} |b\rangle, & T |a\rangle &= \theta(a) \cdot e^{-2\pi i \frac{c}{24}} |a\rangle \end{aligned}$$

How to construct $V(T^2)$

Braiding in TQFT $\cong$ Monodromy in CFT

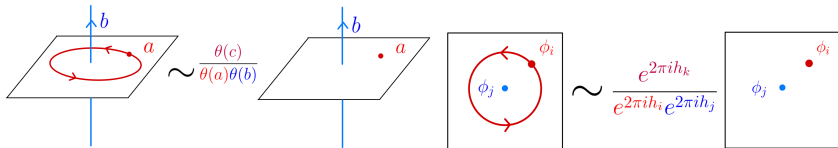
S matrix and OPE:

$$S_{ab} = \frac{1}{\sqrt{\sum_a d_a^2}} \sum_c N_{a\bar{b}}^c \frac{\theta(c)}{\theta(a)\theta(b)} d_c$$

$$\phi_i(z)\phi_j(0) \sim \sum_k C_{ij}^k z^{h_k-h_i-h_j} \phi_k(0) + \dots$$

Consider $z \mapsto z e^{2\pi i}$, then the monodromy phase arises to be

$$e^{2\pi i(h_k-h_i-h_j)} = \frac{e^{2\pi i h_k}}{e^{2\pi i h_i} e^{2\pi i h_j}} \sim \frac{\theta(c)}{\theta(a)\theta(b)}$$



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How to encode time-reversal symmetry

Group action on anyons in general:

Finite group $G \curvearrowright \{\text{Anyons}\} = \mathcal{C}$

→ “representation on anyon”

$$\rho : G \rightarrow \text{Aut}(\mathcal{C})$$

$$g \mapsto \rho_g$$

where $\rho_g : a \mapsto \rho_g(a)$

→ Construct $\bigoplus_{g \in G} \mathcal{C}_g$ such that

$$a_g \otimes b_h \in \mathcal{C}_{gh},$$

with the associativity constraint

$$a_g \otimes (b_h \otimes c_k) \cong (a_g \otimes b_h) \otimes c_k.$$

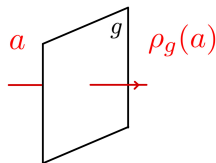


Figure: Group action on anyon a

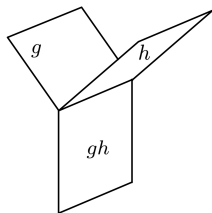


Figure: Branchsheet encoding G

How to encode time-reversal symmetry

$\exists H^3$ obstructions $\rightarrow$ Associativity of G breaks.

$\rightarrow$ Such symmetry is described by 2-group, [2, 6]
i.e., mixed structure of 0-form and 1-form symmetry.

If $e \in H_\rho^3(G, \mathcal{A})$ where $\mathcal{A}$: invertible objects in $\mathcal{C}$ (abelian anyons),

$$a_g \otimes (b_h \otimes c_k) \cong ((a_g \otimes b_h) \otimes c_k) \otimes e(g, h, k).$$

Fact

There is no known example where H^3 obstruction exists in abelian TQFT, but no proof for G in general.

(With an assumption which seems very natural to physicists, it is shown for unitary cases [2].)

How to encode time-reversal symmetry

We define the time-reversal operator $T: \mathcal{C} \rightarrow \mathcal{C}$ by the condition:

$$T: a \mapsto Ta, \text{ such that } T^2 = \text{id}_{\mathcal{C}}, \quad \theta(Ta) = \theta(a)^{-1} \text{ for all } a \in \mathcal{C}.$$

Then, we can define Local Kramers degeneracy: We define a map

$$\eta: \mathcal{C} \rightarrow \{0, \pm 1\} \quad \text{such that} \quad \eta(a) = \begin{cases} \pm 1 & \text{if } a = Ta, \\ 0 & \text{otherwise.} \end{cases}$$

Remark

Here, we simply write $Ta := \rho_g(a)$ where $\rho: \mathbb{Z}_2 = \{1, g\} \rightarrow \text{Aut}(\mathcal{C})$.

How to encode time-reversal symmetry

Known facts for time reversal:

- For necessary conditions for H^3 obstruction to vanish, see [1].
→ In abelian case, they were proved to satisfy [4].
- In abelian cases, H^3 obstruction was proved to vanish [5].
- The nontrivial example exists [1].

Anomaly Inflow:

- From the view point of the anomaly inflow, anomalies in TQFTs with time-reversal symmetry are characterized by

$$Z_{\text{CY}}(\mathbb{RP}^4), Z_{\text{CY}}(\mathbb{CP}^2) \in \{\pm 1\}.$$

- There is a constraint known as the anomaly formula [8, 3]:

$$Z_{\text{CY}}(\mathbb{RP}^4)Z_{\text{CY}}(\mathbb{CP}^2) = \theta_{\mathcal{M}}$$

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Analysis of Crosscap

We define the **Crosscap** state as:

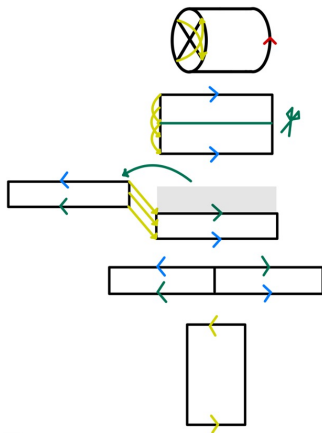
$$\begin{aligned} |\mathbb{C}\mathbb{C}\rangle &= Z(\text{diagram}) \times S^1) \\ &= \sum_a M_a |a\rangle \\ &= \sum_a \eta(a) S |a\rangle, \end{aligned}$$

where M_a and $\eta(a)$ are coefficients.

Then, we have

$$M_a = \langle a | \sum_b \eta(b) S |b\rangle = \sum_b S_{ab} \eta(b)$$

$$\eta(a) = \langle a | S^{-1} |\mathbb{C}\mathbb{C}\rangle$$



Analysis of Crosscap

What is M_a ?:

$$\begin{aligned}M_a &= \langle a | \mathbb{C}\mathbb{C} \rangle \\&= Z(\text{disk with } \times \text{ and } \rightarrow) \times S^1 Z(\text{rectangle with } \times \text{ and } \rightarrow) \times S^1 \\&= Z(\text{rectangle with } \times \text{ and } \rightarrow) \times S^1 \\&= Z(\text{disk with } \times \text{ and } \rightarrow) \times S^1 \\&= \dim V(\mathbb{R}\mathbb{P}^2(a))\end{aligned}$$

Recall

$$\forall \text{ non-orientable surface} = \text{disk with } \times \text{ and } \rightarrow \# \cdots \# \text{disk with } \times \text{ and } \rightarrow$$

M_a should be an integer, but it is not always true !

$\rightarrow M_a \in \mathbb{Z}_{\geq 0}$ is one of the necessary conditions for H^3 obstruction to vanish [1, 4].

Analysis of Crosscap

From topological point of view:

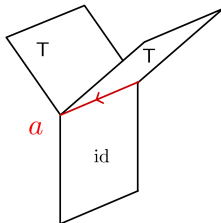
$$Z(\#M) \in V(\partial M) = V(\mathbb{RP}^2)$$

$$Z(\exists M') \in V(\partial M') = V(\mathbb{RP}^2 \sqcup \mathbb{RP}^2)$$

→ It is imposible to realize a physical state on a single $\mathbb{RP}^2$,
but possible on a pair of $\mathbb{RP}^2$!

c.f.) Majorana zero modes

→ So, what is the meaning of a such that $M_a = \dim V(\mathbb{RP}^2(a)) > 0$?
At least in abelian TQFTs [3],

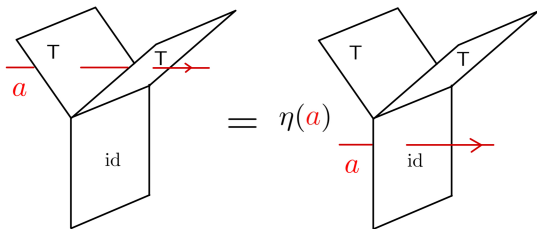


Analysis of Crosscap

What is $\eta(a)$?: $\eta(a)$ can be computed as follows [7]:

$$\begin{aligned}\eta(a) &= \langle a | S^{-1} | \mathbb{C}\mathbb{C} \rangle \\ &= \dots \\ &= Z \left(\frac{S^2(a, \overline{T}a) \times [0, 1]}{(0, x) \sim (1, -x)} \right)\end{aligned}$$

This value is understood as local Kramers degeneracy, or a projective phase arising from T^2 on time-reversal invariant anyons:



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Toward a formulation of non-orientable TQFTs

Reshetikhin–Turaev invariant:

$\langle \rangle : M \rightarrow \mathbb{C}$ where M is an orientable closed three-manifold.

- We can construct Reshetikhin–Turaev TQFT from this invariant.
- Reshetikhin–Turaev TQFT is understood as a mathematical formulation of Chern–Simons theories.
- This invariant is constructed via Lickorish–Wallece theorem.

Lickorish–Wallece theorem:

For any closed connected three-manifold M , there exists a framed link L such that $S_L^3 \cong M$, where S_L^3 is a three-manifold obtained from S^3 via surgery along L .

- Framed link $\sim$ Link with an integer $\in \mathbb{Z}$.
- Surgery $\rightarrow$ See the next slide.

Toward a formulation of non-orientable TQFTs

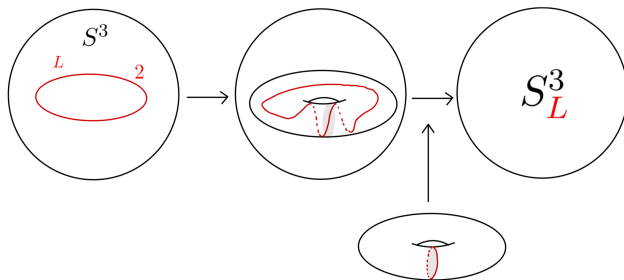


Figure: Rough picture of the Surgery

$$M \cong S_L^3 \rightarrow \langle M \rangle \sim \sum_{a \in \mathcal{C}} \langle L(a) \rangle$$

Here, let us recall

$$\eta(a) = Z \left(\frac{S^2(a, \overline{1a}) \times [0, 1]}{(0, x) \sim (1, -x)} \right) =: Z(X^3(a)).$$

Toward a formulation of non-orientable TQFTs

Surprisingly, X^3 is the building block of non-orientable three-manifolds!

$$\begin{array}{l} S^3 \rightarrow M \xrightarrow{\mathcal{C}} \langle M \rangle \rightarrow \text{TQFT} \\ X^3 \rightarrow N \quad \boxed{\xrightarrow{\mathcal{C}} \langle N \rangle \rightarrow \text{TQFT?}} \end{array}$$

Here, M, N is an arbitrary orientable/non-orientable three-manifold. We need to know what kind of $\mathcal{C}$ is the right data to construct the invariant and TQFTs for non-orientable manifolds.

Physics gives us the clue!

Reference

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Thank you for listening.