

Insights from the crosscap state toward non-orientable TQFTs

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Based on [IO 2509.07368] and [IO 2502.13532],
and on works in progress with Y. Tachikawa and with S. Suzuki.

Motivation

(2+1)D bosonic **Topological quantum field theory (TQFT)** is a QFT which

- depends only on the topology of the spacetime manifolds
- is characterized by anyon data.

Fact: Anyon data $\xleftrightarrow{1:1}$ Modular tensor category $\mathcal{C}$ (MTC)

→ We can construct TQFT mathematically using MTC.

But only on orientable manifolds!

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But only on orientable manifolds!

In general, it is understood that

QFT with time-reversal symmetry = QFT on non-orientable manifolds

→ It is natural to expect that we can construct non-orientable TQFTs.

Question: Is there any implication from physics? → **Yes!**

- TQFT and anyon
- Time reversal on anyon
- Introduction of crosscap
- Insights from crosscap

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Example: $U(1)$ Chern–Simons theory

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int_M A \wedge dA$$

where $k \in \mathbb{Z}$ is a level and A is a $U(1)$ gauge field.

→ $\frac{\delta S_{\text{CS}}}{\delta g_{\mu\nu}} = 0$: independent of metric.

→ $Z[M] := \int \mathcal{D}A \exp(iS_{\text{CS}})$ gives an invariant of M (i.e. TQFT!).

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Physical observable: Wilson loop labeled by **anyon**:

$$W(\gamma, a) = \exp \left(ia \oint_{\gamma} A \right)$$

where γ is a loop in M and $a \in \mathbb{Z}/k\mathbb{Z} \cong \{0, 1, \dots, k-1\}$

→ $\langle W(\gamma, a) \rangle$ is computed as a knot invariant.

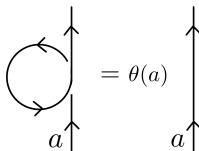
TQFT and anyon

Anyons: Generalized concepts of bosons and fermions.

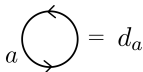
Fusion rule:

$$a \times b = \sum_c N_{ab}^c c, \quad a, b, c \in \{\text{Anyons}\} =: \mathcal{C}, \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}$$

Topological spin: $\theta : \mathcal{C} \rightarrow U(1)$ encoding spin statistics of anyons:


$$\text{Diagram: a vertical line with an upward arrow labeled } a \text{ and a circular loop with two arrows (one clockwise, one counter-clockwise) attached to it.} = \theta(a) \text{ a vertical line with an upward arrow labeled } a$$

Quantum dimension: $d_a \in \mathbb{R}_{\geq 1}$ $a \in \mathcal{C}$, with $\mathcal{D} := \sqrt{\sum_a d_a^2}$


$$\text{Diagram: a circular loop with two arrows (one clockwise, one counter-clockwise).} = d_a$$

Modular S matrix:

$$S_{ab} = \frac{1}{\mathcal{D}} \sum_c N_{ab}^c \frac{\theta(c)}{\theta(a)\theta(b)} d_c = \frac{1}{\mathcal{D}} \text{Diagram: two overlapping circles labeled } a \text{ and } b \text{ with arrows pointing into them from the left.}$$

Fact: Anyon data $\xleftrightarrow{1:1}$ Modular tensor category $\mathcal{C}$ (MTC)

→ MTC encodes data such as

- anyons
- fusion rule
- topological spin
- quantum dimension
- S matrix
- ...

Remark: TQFT is said to be abelian if $a \times b = c$ uniquely for all fusions.

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$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int_M A \wedge dA$$

where $k \in \mathbb{Z}$ is a level and A is a $U(1)$ gauge field.

Input data: $k \in \mathbb{Z}$

Anyons: $a, b, \dots \in \mathbb{Z}/k\mathbb{Z}$

Fusion rule: $a + b \in \mathbb{Z}/k\mathbb{Z}$

Topological spin: $\theta(a) := \exp(\pi i a^2 / k)$

Example: $U(1)^N$ Chern–Simons theory

$$S_{\text{CS}}[A] = \frac{1}{4\pi} \int_M K_{IJ} A^I \wedge dA^J$$

where $K = (K_{IJ}) \in \text{Mat}_N(\mathbb{Z})$ is a K matrix and $\{A^I\}_{I=1,\dots,N}$ are $U(1)$ gauge fields.

Input data: $k \in \mathbb{Z} \rightarrow K \in \text{Mat}_N(\mathbb{Z})$

Anyons: $a, b, \dots \in \mathbb{Z}/k\mathbb{Z} \rightarrow a, b, \dots \in \mathbb{Z}^N / K\mathbb{Z}^N$

Fusion rule: $a + b \in \mathbb{Z}/k\mathbb{Z} \rightarrow a + b \in \mathbb{Z}^N / K\mathbb{Z}^N$

Topological spin: $\theta(a) := \exp(\pi i a^2 / k) \rightarrow \theta(a) := \exp(\pi i a^T K^{-1} a)$

Example: Toric code $\sim K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} U(1)^2$ Chern–Simons theory

$\{1, e, m, f\}$ with $e \times e = m \times m = f \times f = 1$, $e \times m = f$

Input data: $K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

Anyons: $\left\{ 1 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, e := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, m := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, f := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \in \mathbb{Z}^2 / K\mathbb{Z}^2$

Fusion rule: $e \times m = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = f$

Topological spin: $\theta(f) = \exp \left(\pi i (1, 1) \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = -1$

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Time reversal on anyon

With time-reversal, we have

$$\frac{1}{4\pi} \int_M K^{IJ} A_I \wedge dA_J \mapsto \frac{1}{4\pi} \int_{\textcolor{red}{-M}} K^{IJ} A_I \wedge dA_J = \frac{1}{4\pi} \int_M -K^{IJ} A_I \wedge dA_J$$

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Example: Consider Toric code and $T \in \text{Mat}_2(\mathbb{Z})$ as follows:

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{satisfying } T^T K T = -K$$

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Then we obtain

$$\begin{aligned} S &= \frac{1}{4\pi} \int_{-M} (A_1, A_2) \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} dA_1 \\ dA_2 \end{pmatrix} \\ &= \frac{1}{4\pi} \int_M (A_1, A_2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} dA_1 \\ dA_2 \end{pmatrix} \\ &= \frac{1}{4\pi} \int_M (A_2, -A_1) \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} dA_2 \\ -dA_1 \end{pmatrix}. \end{aligned}$$

Time reversal on anyon

Observation: Time-reversal induces $(A_1, A_2) \mapsto (A_2, -A_1)$

$$\mathsf{T} : \exp \left(i \oint_{\gamma} A_1 \right) \mapsto \exp \left(i \oint_{\gamma} A_2 \right).$$

Or more simply put, T permutes e and m

$$\mathsf{T}e = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} = m$$

How do we describe time reversal on anyons categorically and what additional data do we need?

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How do we describe time reversal on anyons categorically and what additional data do we need? $\rightarrow$ Symmetry fractionalization

Time reversal on anyon

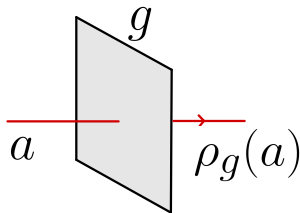
Unitary group action on anyons:

Finite group $G \curvearrowright \{\text{Anyons}\} = \mathcal{C}$

→ Construct $\bigoplus_{g \in G} \mathcal{C}_g$ s.t. $a_g \otimes b_h \in \mathcal{C}_{gh}$,
with the associativity constraint

$$a_g \otimes (b_h \otimes c_k) \cong (a_g \otimes b_h) \otimes c_k.$$

Here, a_g is a nongenuine line operator
bounding domain wall for g .



Time reversal on anyon

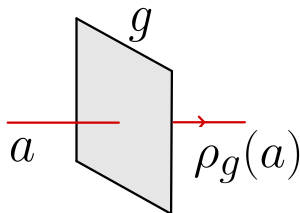
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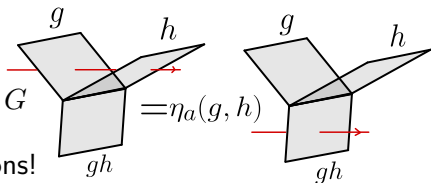


Symmetry fractionalization:

We have additional data which looks like
projective phases

$$\eta_a(g, h) \in U(1) \quad \text{for } a \in \mathcal{C}, g, h \in G$$

→ Basic data to evaluate partition functions!



Time reversal on anyon

Remark: It is not always possible to construct such an extension.

→ $e \in H^3(G, \mathcal{A})$ obstructs the associativity as

$$a_g \otimes (b_h \otimes c_k) \cong ((a_g \otimes b_h) \otimes c_k) \otimes e(g, h, k).$$

Here $\mathcal{A}$ is the set of abelian anyons i.e. $e(g, h, k)$ is an abelian anyon.

→ Also, we cannot define symmetry fractionalization with the obstruction.

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When G is time reversal, i.e. anti-unitary $\mathbb{Z}_2$, it is unknown that such an extension $\bigoplus_{g \in \mathbb{Z}_2^\Gamma} \mathcal{C}_g$ exists. **However**, H^3 obstruction can be defined with which we cannot define symmetry fractionalization $\eta_a(\mathbb{T}, \mathbb{T}) =: \eta(a)$ [Barkeshli et al. 1410.4540, 1612.07792].

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- H^3 obstruction vanishes for all abelian TQFTs [IO 2509.07368].
- Nontrivial examples are known for nonabelian TQFTs [Barkeshli et al. 1706.09464].

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Introduction of crosscap

Recall: Algebraically, $N_{ab}^c = \sum_d S_{ad} S_{bd} \overline{S_{cd}} / S_{0d} \in \mathbb{Z}_{\geq 0}$ is nontrivial.
From TQFT perspectives, $N_{ab}^c = \dim V(S^2(a, b, \bar{c})) \in \mathbb{Z}_{\geq 0}$ is natural.

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Given $\mathcal{C}$ and time-reversal $\mathbb{Z}_2^T \curvearrowright \mathcal{C}$ without H^3 -obstruction:

$$M_a := \sum_{b=\text{T}b} S_{ab} \eta(b)$$

RHS is purely determined with MTC. Surprisingly, we have the followings:

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RHS is purely determined with MTC. Surprisingly, we have the followings:

Fact:

- $M_a \in \mathbb{Z}_{\geq 0}$ for every known example.
 - $M_a \in \mathbb{Z}_{\geq 0}$ is shown for all abelian TQFTs [IO 2502.13532].
- These are highly nontrivial and give some implications.

Introduction of crosscap

Axiom.I: Assign $V(\Sigma)$ to closed surface Σ .

Axiom.II: $V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2) \rightarrow V(\emptyset) = \mathbb{C}$.

Axiom.III: $Z(M) \in V(\partial M) \rightarrow Z(M) \in \mathbb{C}$ for closed M .

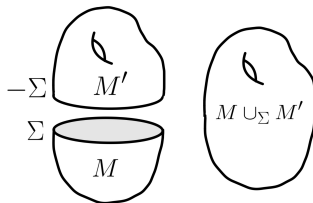
Axiom.IV: $V(-\Sigma) = V(\Sigma)^*$.

Gluing:

$$|\psi\rangle := Z(M) \in V(\Sigma)$$

$$\langle\psi'| := Z(M') \in V(\Sigma)^*$$

$$\langle\psi'|\psi\rangle = Z(M \cup_{\Sigma} M') \in V(\emptyset) = \mathbb{C}$$



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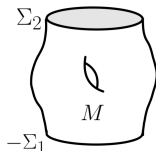
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Cobordism:

$$\partial M = -\Sigma_1 \sqcup \Sigma_2,$$

$$Z(M) \in V(\Sigma_1)^* \otimes V(\Sigma_2), \text{ i.e. } Z(M) : V(\Sigma_1) \rightarrow V(\Sigma_2).$$

$$Z(M) = \sum_{a,b} U_{ab} |\psi_{\Sigma_2,a}\rangle \otimes \langle \psi_{\Sigma_1,b}|.$$



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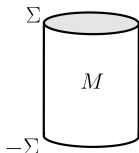
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Identity cobordism:

$$M = \Sigma \times I,$$

$$\partial M = -\Sigma \sqcup \Sigma,$$

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Dimension formula:

$$\begin{aligned} Z(\Sigma \times S^1) &= \text{Tr}[Z(\Sigma \times I)] \\ &= \text{Tr}\left(\sum_a |\psi_{\Sigma,a}\rangle \langle \psi_{\Sigma,a}|\right) \\ &= \dim V(\Sigma) \end{aligned}$$

Introduction of crosscap

Recall:

- $Z(M) \in V(\partial M)$
- $Z(\Sigma \times S^1) = \dim V(\Sigma)$

→ We can construct bases on $V(T^2)$ as follows:

Introduction of crosscap

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→ We can construct bases on $V(T^2)$ as follows:

- **Constructing a basis:**

$$|a\rangle := Z(D^2(a) \times S^1) = Z(\text{disk with red dot} \times S^1) = Z(\text{torus with red loop}) \in V(T^2),$$

$$\rightarrow V(T^2) = \text{span}_{\mathbb{C}} \{|a\rangle \mid a \in \mathcal{C}\}.$$

- **Another basis via modular S :**

$$S|a\rangle := Z(S^1 \times D^2(a)) \in V(T^2),$$

$$\rightarrow V(T^2) = \text{span}_{\mathbb{C}} \{S|a\rangle \mid a \in \mathcal{C}\}.$$

Remark: We have two bases to expand torus partition functions $\{\chi_a(\tau)\}_{a \in \mathcal{C}}$ and $\{\chi_a(-1/\tau)\}_{a \in \mathcal{C}}$.

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Insights from crosscap

We can compute M_a as follows:

$$\begin{aligned} M_a &= \langle a | \mathbb{C}\mathbb{C} \rangle \\ &= Z(\text{cap} \times S^1) Z(\text{crosscap} \times S^1) \\ &= Z(\text{crosscap} \times S^1) \\ &= Z(\text{cap} \times S^1) \\ &= \dim V(\mathbb{R}P^2(a)) \end{aligned}$$

where we used $Z(\Sigma \times S^1) = \dim V(\Sigma)$ for closed surfaces.

If TQFT exists, $M_a \in \mathbb{Z}_{\geq 0}$ has a natural interpretation.

→ Suggestion for the existence of TQFTs on non-orientable manifolds.

Insights from crosscap

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→ Suggestion for the existence of TQFTs on non-orientable manifolds.

→ How about $\eta(a)$?

Reshetikhin–Turaev invariant:

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- Reshetikhin–Turaev TQFT is understood as a mathematical formulation of Chern–Simons theories.
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Lickorish–Wallace theorem:

For any closed connected **orientable** three-manifold M , there exists a framed link L such that $S_L^3 \cong M$, where S_L^3 is a three-manifold obtained from S^3 via surgery along L .

- Framed link $\sim$ Link with an integer $\in \mathbb{Z}$.
- Surgery $\rightarrow$ See the next slide.

Insights from crosscap

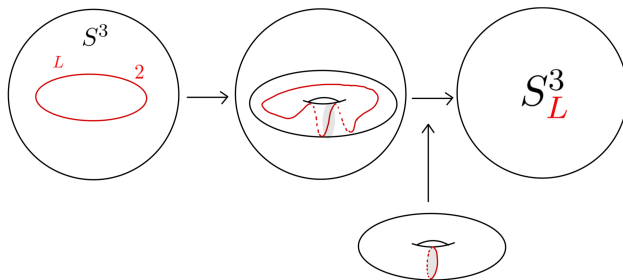


Figure: Rough picture of the Surgery

$$M \cong S^3_{L=\bigsqcup_i L_i} \rightarrow \langle M \rangle \sim \sum_{a,b,\dots \in \mathcal{C}} d_a d_b \cdots \langle L(a,b,\dots) \rangle$$

Insights from crosscap

Get back to our set up,

We can compute $\eta(a)$ as follows: [Tachikawa-Yonekura 1611.01601]

$$\begin{aligned}\eta(a) &= \langle a | S^{-1} | \mathbb{C}\mathbb{C} \rangle \\ &= \dots \\ &= Z \left(\frac{S^2(a, \overline{1a}) \times [0, 1]}{(0, x) \sim (1, -x)} \right) \\ &=: Z(X^3(a))\end{aligned}$$

Recall: $\eta(a)$ is understood as symmetry fractionalization.

But it has another crucial implication.

Insights from crosscap

X^3 is the building block of non-orientable manifolds ! [Lickorish 1963]

Future work:

- $\forall M:\text{orientable}, S^3 \xrightarrow{\exists L} S_L^3 \cong M \xrightarrow{\mathcal{C}} \langle M \rangle \rightarrow \text{TQFT}.$
- $\forall M:\text{non-orientable}, X^3 \xrightarrow{\exists L} X_L^3 \cong M \xrightarrow{\mathbb{Z}_2^T \curvearrowright \mathcal{C}} \langle M \rangle \rightarrow \text{TQFT}?$

Here, the red part remains to be understood.

Physics gives us the clue!

Question: Is there any implication from physics? $\rightarrow$ Yes!

Thank you for listening.