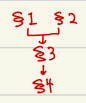


Introduction to SymTFT and Condensation Defects via $\mathbb{Z}_2$ gauge theory.

Contents

- §1. Quick review of 2+1 D TQFT.
- §2. Duality web on 1+1 D QFT.
- §3. SymTFT with $\mathbb{Z}_2$ gauge theory
- §4. Condensation defects with $\mathbb{Z}_2$ gauge theory



Ref

- §1. • 2205.05565
 - "Topological Quantum"
- §2.3. • "Lecture on anomalies and topological phases (2019)"
 - Refs in 2601.09067
 - 2107.13091
- §4. • 2204.02407

§1 Quick review of 2+1 D TQFT

Fact

2+1 D TQFTs are described by Modular Tensor Cat (MTC).

Basic data of MTC

- Anyon : $a, b, c, \dots \in \{\text{Anyon}\} =: \mathcal{C}$
- Fusion : $a \times b = \sum_c N_{ab}^c c \quad \text{w/ } N_{ab}^c \in \mathbb{Z}_{\geq 0}$
- Top spin : $\theta(a) \in U(1) \quad \text{w/ } a \in \mathcal{C}$
- Braciding : $B(a,b) \in U(1) \quad \text{w/ } a, b \in \mathcal{C}$
- Quantum dim : $d_a \in \mathbb{R}_{\geq 1} \quad \text{w/ } a \in \mathcal{C}$
- Top. q. dim : $D := \sqrt{\sum_a d_a^2}$
- S mat : $S_{ab} := \frac{1}{D} \sum_c N_{ab}^c \frac{\theta(c)}{\theta(a)\theta(b)} d_c$
- Chiral central charge : $\frac{1}{D} \sum_a d_a^2 \theta(a) = e^{2\pi i c/24}$

Rmk

- $\exists! g \in \mathcal{C} \text{ s.t. } \forall a \in \mathcal{C}, 1 \times a = g$
 - $\forall a \in \mathcal{C}, \exists! \bar{a} \in \mathcal{C} \text{ s.t. } a \times \bar{a} = 1$
- TQFT is abelian iff $a \times b = c \quad \forall a, b \in \mathcal{C}, \Leftrightarrow d_a = 1 \quad \forall a \in \mathcal{C}$

Fact

- $\mathcal{C}$: abelian $(\Leftrightarrow N_{ab}^c = \delta_{c, a \times b})$
- $B(a,b) := \frac{\theta(a \times b)}{\theta(a)\theta(b)}$
- $S_{ab} := \frac{1}{|T|^2} \sum_c \delta_{c, a \times b} \frac{\theta(c)}{\theta(a)\theta(b)} d_c$
 $= \frac{1}{|T|^2} B(a, \bar{b})$

Fact

- $SL(2, \mathbb{Z}) \sim \mathcal{V}(T^2)$
- $S_{ab} = \frac{1}{D} \sum_c N_{ab}^c \frac{\theta(c)}{\theta(a)\theta(b)} d_c \quad T_{ab} = e^{-2\pi i c/24} \cdot \theta(a) S_{ab}$

E.g.

$\mathbb{Z}_2$ gauge theory = Toric Code (TC)

Anyon: $C = \{1, e, m, f\}$

Fusion: $e^2 = m^2 = f^2 = 1$ $e \times m = f$ ($\Rightarrow a = \bar{a} \forall a \in C$)

Top spin: $\theta(1) = \theta(e) = \theta(m) = 1$ $\theta(f) = -1$

Braid: $B(e, m) = \frac{\theta(e \times m)}{\theta(e) \theta(m)} = -1$

g dim: $d_a = 1 \forall a \in C$

Top. g dim: $D = 2$

S mat: $S_{ab} = \frac{1}{2} \cdot \frac{\theta(a \times b)}{\theta(a) \theta(b)}$

CC: $c = 0 \pmod{8}$

Fact

$$a \bigcup_b = \frac{S_{ab}}{S_{ib}} \bigcup_b \quad \text{or} \quad S_{ia} = \frac{1}{D} \cdot \sum_c N_{ac}^i \cdot \frac{\theta(c)}{\theta(a) \cdot 2} \cdot d_c = \frac{d_a}{D}$$

Obs

$$\begin{aligned} a \bigcup_b &= \frac{S_{ab}}{S_{ib}} \bigcup_b \\ &= S_{ab} \cdot \left(\frac{P}{d_b} \right) \cdot d_b \\ &= D \cdot S_{ab} \quad \Rightarrow \quad S_{ab} = \frac{1}{D} \cdot \bigcup_b \end{aligned}$$

Cor

$$\begin{aligned} &\left\{ \frac{S_{ax}}{S_{ix}} \right\}_{a \in C} \text{ forms a fusion ring.} \\ \odot \quad i \bigcup_x &= \frac{S_{ix}}{S_{ix}} \frac{S_{ax}}{S_{ix}} \bigg|_x \\ &= \sum_c N_{ic}^a \cdot c \bigg|_x \\ &= \sum_c N_{ic}^a \frac{S_{cx}}{S_{ix}} \bigg|_x \end{aligned}$$

Prop

$$N_{ab}^c = \sum_x \frac{S_{ax} S_{bx} \bar{S}_{cx}}{S_{ix}} \quad (\text{Verlinde formula})$$

$\odot$

$$\frac{S_{ax}}{S_{ix}} \frac{S_{bx}}{S_{ix}} = \sum_c N_{ab}^c \frac{S_{cx}}{S_{ix}}$$

$$\begin{aligned} \Rightarrow \sum_x \frac{S_{ax} S_{bx} \bar{S}_{cx}}{S_{ix}} &= \sum_{x,d} S_{ax} \bar{S}_{dx} N_{ab}^c \\ &= \sum_d (S S^\dagger)_{cd} N_{ab}^c = N_{ab}^c \end{aligned}$$

Fact

$$\begin{aligned} \bullet V(T^2) &\cong \mathbb{C}^{|C|} \cong \text{span}_{\mathbb{C}} \{ |a\rangle \mid a \in C \} = \text{span}_{\mathbb{C}} \{ \bigcirc, \bigcirc, \bigcirc, \dots \} \\ &\cong \text{span}_{\mathbb{C}} \{ |S(a)\rangle \mid a \in C \} = \text{span}_{\mathbb{C}} \{ \bigcirc, \bigcirc, \dots \} \\ \text{w/ } |a\rangle &:= A(x) |0\rangle = \mathbb{Z} \left(\bigcirc \right) = \mathbb{Z} (D^2 a \times S^1) \\ |S(a)\rangle &:= A(x) (S^1 a) = \mathbb{Z} \left(\bigcirc \right) = \mathbb{Z} (S^1 \times D^2 a) \\ \bullet A(x) |b\rangle &= \sum_c N_{ab}^c |c\rangle \end{aligned}$$

Prop

$$A(x) |b\rangle = \frac{S_{ab}}{S_{ib}} |b\rangle \quad \bigcirc = \frac{S_{ab}}{S_{ib}} \bigcirc$$

$\odot$ LHS

$$\begin{aligned} &= S(A(x) |g^+ \rangle |b\rangle) \\ &= S \left(\sum_c A(x) |g^+ \rangle_{bc} |c\rangle \right) \\ &= S \left(\sum_{c,d} (S^+)^d_{bc} N_{ac}^d |d\rangle \right) \\ &= \sum_{c,d,e} \underbrace{S_{ad} N_{ac}^d \cdot (S^+)^e_{bc}}_{= S_{ab} \frac{S_{ee}}{S_{ie}} \quad \odot \text{ Verlinde formula}} |e\rangle = \text{RHS.} \end{aligned}$$

E.g.

$$E(x) |e\rangle = |e\rangle, \quad E(x) |m\rangle = |f\rangle, \quad E(y) |m\rangle = -|m\rangle, \dots$$

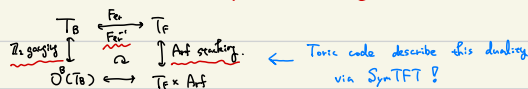
Summary

2+1 D TQFT is described by anyon data, which is encoded by MTC.

§ 2. Duality web on 1+1 D QFT.

Claim

Duality Web on 1+1 D thry wr $\mathbb{Z}_2$ sym.



Notation

- T_B : Bosonic thry coupled to $\mathbb{Z}_2$ back ground.
- T_F : Fermionic " dep on spin str.
- $a \in H^1(\Sigma, \mathbb{Z}_2)$: $\mathbb{Z}_2$ back ground.
- $\eta \in \text{Spin}(\Sigma)$: spin str on Σ .
- $Z_B[a]$: part. fun of T_B .

O^B : Gauging

Recall

Gauging Lie grp G .

$$G \xrightarrow{A} e^{iA} \xrightarrow{A \in \mathbb{R}^n} g.$$

A : η valued 1-form

① "Couple" background gauge fld A to the action.

$$Z_B[A] := \int \mathcal{D}\phi e^{iS[A, \phi]}$$

② Make it dynamical.

i.e. $Z_{T_B/G} := \int \mathcal{D}A Z_B[A]$ is very complicated

Fact

$$G = \mathbb{Z}_2,$$

$$\Rightarrow \text{Part fun } Z_B[a] \quad a \in H^1(\Sigma, \mathbb{Z}_2)$$

$$\text{Path int. } \int \mathcal{D}A \sim \frac{1}{|H^1(\Sigma, \mathbb{Z}_2)|^{\frac{1}{2}}} \sum_{a \in H^1(\Sigma, \mathbb{Z}_2)} \int \mathcal{D}\phi e^{iS[A, \phi]}$$

$$\therefore Z_{T_B/\mathbb{Z}_2} = \frac{1}{|H^1(\Sigma, \mathbb{Z}_2)|^{\frac{1}{2}}} \sum_a Z_B[a]$$

Eg

$$\Sigma = T^2 \quad \therefore H^1(T^2; \mathbb{Z}_2) \cong \{0, z, \bar{z}, z + \bar{z}\}$$

$$Z_{T_B/\mathbb{Z}_2} = \frac{1}{2} (\square + \square + \square + \square)$$

Remark

We can refine it by considering dual sym.

Def

$$Z_{O^B(T_B)}[a] := \frac{1}{|H^1(\Sigma, \mathbb{Z}_2)|^{\frac{1}{2}}} \sum_b (-1)^{\sum_b} Z_B[b]$$

by fld of dual $\mathbb{Z}_2$ sym.

$T_B \cdots$
 $\downarrow$
 $O^B(T_B) \cdots$

Obs

$$(O^B)^2 = 1 \odot Z_{O^B(T_B)}[a] = \frac{1}{|H^1(\Sigma, \mathbb{Z}_2)|^{\frac{1}{2}}} \sum_b (-1)^{\sum_b} \sum_c (-1)^{\sum_c} Z_B[c]$$

$$= Z_B[a].$$

Fer: Fermionization

$\text{Fer}(T_B)$: thry which depends on the spin str η on the sptime.

Def

Spin str η on Σ

(*) The choice of $\eta \in C^1(\Sigma, \mathbb{Z}_2)$ st. $w_2 = S\eta$

Here w_2 : 2nd Stiefel-Whitney class, $\in H^2(\Sigma, \mathbb{Z}_2)$

(*) The choice of lift $\text{Pospic}(\eta) \rightarrow \text{Pospic}(\eta)$ obstruction to the existence of the spin str.

Prop

$\text{Spin}(I)$ forms a torsion over $H(\Sigma, \mathbb{Z}_2)$

i.e. $\forall \eta, \eta' \in \text{Spin}(I)$, $\eta - \eta' \in H(\Sigma, \mathbb{Z}_2)$

⊙

$\eta, \eta' \in \text{Spin}(I) \Rightarrow \omega_2 = S\eta$, $\omega_2 = S\eta'$

$\therefore S(\eta - \eta') = 0$ i.e. $\eta - \eta' \in \text{Im } S$

$\tilde{\eta} = \eta + S\lambda \Leftrightarrow \eta - \tilde{\eta} = S\lambda$ $S(\eta - \tilde{\eta}) = 0$ trivially

$\therefore \eta - \eta' \in \text{Im } \frac{S}{\text{Ker } S} = H(\Sigma, \mathbb{Z}_2)$

Fact

$\{\text{spin str } \eta \text{ on } \Sigma\} \xrightarrow{\text{def}} \{ \tilde{\eta}_1 : H(\Sigma, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2 \text{ s.t. } \tilde{\eta}_1(a+b) = \tilde{\eta}_1(a) + \tilde{\eta}_1(b) + S_{\text{sub}} \}$

Prop

$\tilde{\eta}_{\eta+\lambda}(b) = \tilde{\eta}_\eta(b) + S_{\text{sub}} \quad \forall a, b \in H(\Sigma, \mathbb{Z}_2)$

⊙

$r_a(x) := \tilde{\eta}_{\eta+\lambda}(x) - \tilde{\eta}_\eta(x)$

then $r_a(x+\lambda) = r_a(x) + r_a(\lambda)$

w/ non-deg of S_{sub} , $\exists z \in H(\Sigma, \mathbb{Z}_2) \setminus \{0\}$ s.t. $Sz = x \quad \forall x \in H(\Sigma, \mathbb{Z}_2)$

...

Question

How to define Fermionization

- $\bullet Z_{T_0}[a] : \text{bos theory coupled to bgfd } a \in H(\Sigma, \mathbb{Z}_2)$
- $\bullet \tilde{\eta}_1 : H(\Sigma, \mathbb{Z}_2) \rightarrow \mathbb{Z}_2 : \text{quad form which is 1 to 1 to spin str } \eta$

Def

$Z_{\text{Fer}(T_0)}[\eta] := \frac{1}{|H(\Sigma, \mathbb{Z}_2)|^k} \sum_a (-1)^{\tilde{\eta}_1(a)} Z_{T_0}[a]$

$T_0 \rightarrow T_F$
.....

$Z_{\text{Fer}^*(T_F)}[a] := \frac{1}{|H(\Sigma, \mathbb{Z}_2)|^k} \sum_\eta (-1)^{\tilde{\eta}_1(\eta)} Z_{T_F}[\eta]$

$T_0 \leftarrow T_F$
.....

Cor

$Z_{\text{Fer}^*(T_0)}[\eta] = Z_{T_0}[a]$

Def

$Z_{\text{Art}}[\eta] := \frac{1}{|H(\Sigma, \mathbb{Z}_2)|^k} \sum_a (-1)^{\tilde{\eta}_1(a)} \in \mathbb{Z}_2$

Rmk

- $\bullet Z_{\text{Art}}[\eta]$ can be interpreted as a Fermionization of trivial theory.
- $\bullet$ Physically, Art inv is a low energy lim of the Kosterlitz chain.
- $\bullet$ Mathematically, Art inv gives a homomorphism $SL_2^{\text{spin}}(\mathbb{C}, \mathbb{Z}_2) \cong \mathbb{Z}_2$

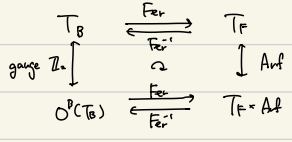
Prop

$Z_{\text{Fer} \circ O^0 \circ \text{Fer}^*(T_F)}[\eta] = Z_{T_F}[\eta] Z_{\text{Art}}[\eta]$

⊙

Easy to show by def.

$\hookrightarrow$ We obtain the duality web :



Rmk

$(O^0)^2 = 1 \Leftrightarrow (Z_{\text{Art}}[\eta])^2 = 1$

§3 SymTFT with $\mathbb{Z}_2$ gauge theory

Slogan

SymTFT: Sym in d-dim thng can be described by d+1 D TQFT

→ See "Module Category and Symmetry: TFT" for general discussions

Fact

1+1 D Duality Web in §2 $\leftrightarrow$ SymTFT: Toric code

i.e. $\mathcal{Z}(\text{Vec } \mathbb{Z}_2) = \text{Toric code MTC}$.

Recall

- Fourier transf in QM

$$\mathcal{F}(\psi)(p) = \frac{1}{\sqrt{2\pi}} \int dx e^{-ipx} \psi(x)$$

- Gauging in 1+1 D

$$\mathcal{Z}^{\text{orb}}(\mathcal{T}_0)[A] = \frac{1}{|H^1(\mathbb{Z}_2, \mathbb{Z})|^{1/2}} \sum_b (-1)^{S_{\text{orb}}} \mathcal{Z}_{\mathcal{T}_0}[b]$$

- Fermionization in 1+1 D

$$\mathcal{Z}_{\text{Ferm}}(\mathcal{T}_0)[\eta] = \frac{1}{|H^1(\mathbb{Z}_2, \mathbb{Z})|^{1/2}} \sum_a (-1)^{Z_1(a)} \mathcal{Z}_{\mathcal{T}_0}[a]$$

Question

Can we interpret $\mathcal{O}^{\text{b}}, \text{Fer}$ as a change of basis?

Answer

$$\langle e, a | \mathcal{T}_0 \rangle := \mathcal{Z}_{\mathcal{T}_0}[a]$$

$$\mathcal{Z}^{\text{orb}}(\mathcal{T}_0)[a] = \langle m, a | \mathcal{T}_0 \rangle$$

$$= \frac{1}{|H^1(\mathbb{Z}_2, \mathbb{Z})|^{1/2}} \sum_b \langle a, a | e, b \rangle \langle e, b | \mathcal{T}_0 \rangle$$

$$= \frac{1}{|H^1(\mathbb{Z}_2, \mathbb{Z})|^{1/2}} \sum_b (-1)^{S_{\text{orb}}} \mathcal{Z}_{\mathcal{T}_0}[b]$$

$$\mathcal{Z}_{\text{Ferm}}(\mathcal{T}_0)[\eta] = \langle \eta | \mathcal{T}_0 \rangle$$

$$= \frac{1}{|H^1(\mathbb{Z}_2, \mathbb{Z})|^{1/2}} \sum_a \langle \eta | e, a \rangle \langle e, a | \mathcal{T}_0 \rangle$$

$$= \frac{1}{|H^1(\mathbb{Z}_2, \mathbb{Z})|^{1/2}} \sum_a (-1)^{Z_1(a)} \mathcal{Z}_{\mathcal{T}_0}[a]$$

$$\text{or } V(\Sigma) \cong \text{span}_{\mathbb{C}} \{ |e, a\rangle \mid a \in H^1(\mathbb{Z}_2, \mathbb{Z}) \}$$

$$\cong \text{span}_{\mathbb{C}} \{ |m, a\rangle \mid a \in H^1(\mathbb{Z}_2, \mathbb{Z}) \}$$

$$\cong \text{span}_{\mathbb{C}} \{ |\eta\rangle \mid \eta \in \text{Spin}(\Sigma^1) \}$$

Next
let us construct
these bases.

$V(\Sigma)$: Hilb sp assigned by 2+1 D TQFT w/ Toric code.

Claim

2+1 D w/ sym: fusion cat $\mathcal{C}$

$\leftrightarrow$ SymTFT: MTC = $\mathcal{Z}(\mathcal{C})$ Drinfeld center.

Fact

Given MTC $\mathcal{C}$,

$\exists \mathcal{F}$: fusion cat s.t. $\mathcal{Z}(\mathcal{F}) \Leftrightarrow \exists \mathcal{A}$: Lagrangian alg in $\mathcal{C}$.

$\downarrow$
 $\mathcal{C}$: abelian

$\mathcal{L}$: Lagrangian subgroup

Remark

For even fn $\psi(x) = \psi(-x)$

$$\mathcal{F}^2(\psi)(x) = \psi(x) \quad \text{i.e. } \mathcal{F}^2 = 1.$$

$\leftrightarrow$ Compatible w/ $(\mathcal{O}^{\text{b}})^2 = 1$.

Recall

Fourier transf = change of basis.

$$\text{i.e. } \mathcal{F}(\psi)(p) = \langle p | \psi \rangle$$

$$= \int dx \langle p | x \rangle \langle x | \psi \rangle$$

$$= \int dx \frac{e^{-ipx}}{\sqrt{2\pi}} \psi(x)$$

Def

For abelian TQFT $\mathcal{C}$, L is a lagrangian subgp

- $\Leftrightarrow$ {
- $\bullet \theta(a) = 1 \quad \forall a \in L$
 - $\bullet B(a, b) = 1 \quad \forall a, b \in L$
 - $\bullet |L|^2 = |\mathcal{C}|$
 - $\bullet \forall a \in \mathcal{C} \setminus L, \exists b \in L \text{ st. } B(a, b) \neq 1.$
- indep of triangulation*
maximal.

Cor

$L = \{1, e, f, 1, m\}$ ($\{1, f\}$: odd) are lagrangian

☺ For $\{1, e\}$,

- $\bullet \theta(1) = \theta(e) = 1$
- $\bullet B(e, e) = \frac{\theta(e, e)}{\theta(e)\theta(e)} = \frac{1}{1 \cdot 1} = 1 \quad e \in \mathcal{C} \dots$
- $\bullet |1, e\rangle|^2 = |\mathcal{C}| = 4$
- $\bullet \mathcal{C} \setminus L = \{m, f\}, B(e, m) = B(e, f) = -1$

Construction of gapped bdy

Input: Toric code TQFT $\leftarrow$ Realized as $U(1)^2$ Chern-Simons $\sim K \circ (a^*)$

Wilson loop op: $\{1, E(a), M(a), F(a)\}$ satisfying

$$(*) \begin{cases} E(a+b) = E(a)E(b), & E(a)^\dagger = E(a) \quad \forall a, b \in H^1(\Sigma, \mathbb{Z}) \\ M(a+b) = M(a)M(b), & M(a)^\dagger = M(a) \quad \forall a, b \in H^1(\Sigma, \mathbb{Z}) \\ F(a) := E(a)M(a) \\ E(a)M(b) = (-1)^{S_{\text{aub}}} M(b)E(a) \quad \forall a, b \in H^1(\Sigma, \mathbb{Z}) \\ E(a) = M(a) = F(a) = 1 \end{cases}$$

Output: Three kinds of bases corresponding to e, m, f

$$\begin{aligned} V(\Sigma) &\cong \text{span}_{\mathbb{C}} \{ |e, a\rangle \mid a \in H^1(\Sigma, \mathbb{Z}) \} \\ &\cong \text{" } \{ |m, a\rangle \mid \text{" } \} \\ &\cong \text{" } \{ |f, a\rangle \mid \text{" } \} \end{aligned}$$

Prop

- $\bullet E(a)^2 = M(a)^2 = F(a)^2 = 1$ ($\leftarrow$ fusion rule of $\{(e, m, f)\}$)
- $\bullet F(a+b) = (-1)^{S_{\text{aub}}} F(a)F(b)$
- ☺ Easy to show wr $\mathcal{C}^*$

Basis w.r.t $E(a)$

Define $|e, 0\rangle$ by the following condition

$$E(a)|e, 0\rangle = |e, 0\rangle \quad \forall a \in H^1(\Sigma, \mathbb{Z})$$

and define $|e, a\rangle = M(a)|e, 0\rangle$

$\Rightarrow |e, b\rangle$ is an eigenvector of $E(a)$

$$\begin{aligned} E(a)|e, b\rangle &= E(a)M(b)|e, 0\rangle \\ &= (-1)^{S_{\text{aub}}} M(b)E(a)|e, 0\rangle \\ &= (-1)^{S_{\text{aub}}} M(b)|e, 0\rangle \\ &= (-1)^{S_{\text{aub}}} |e, b\rangle \end{aligned}$$

Prop wr $\dim V(\Sigma) = |H^1(\Sigma, \mathbb{Z})|$

- $\bullet |e, 0\rangle$ always exists uniquely up to phase.
- $\bullet \{ |e, a\rangle \}_{a \in H^1(\Sigma, \mathbb{Z})}$ is an orthogonal basis of $V(\Sigma)$
- ☺ Using wnidg of S_{aub} many times.

$$\leadsto V(\Sigma) \cong \text{span}_{\mathbb{C}} \{ |e, a\rangle \mid a \in H^1(\Sigma, \mathbb{Z}) \}$$

Basis w.r.t $M(a)$

Define $|m, 0\rangle$ by $M(a)|m, 0\rangle = |m, 0\rangle \quad \forall a \in H^1(\Sigma, \mathbb{Z})$

$|m, a\rangle$ by $|m, a\rangle := E(a)|m, 0\rangle$

$$\leadsto V(\Sigma) \cong \text{span}_{\mathbb{C}} \{ |m, a\rangle \mid a \in H^1(\Sigma, \mathbb{Z}) \}$$

Normalization

$$\langle e, o | m, o \rangle = 1$$

$$\langle e, a | e, b \rangle = |H(\mathbb{C}, \mathbb{Z}_2)|^{\frac{1}{2}} \delta_{ab}$$

$$\langle m, a | m, b \rangle = |H(\mathbb{C}, \mathbb{Z}_2)|^{\frac{1}{2}} \delta_{ab}$$

$$\text{id}_{V(\Sigma)} = \frac{1}{|H(\mathbb{C}, \mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_a |e, a\rangle \langle e, a| = \frac{1}{|H(\mathbb{C}, \mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_a |m, a\rangle \langle m, a|$$

Cor

$$\langle e, a | m, b \rangle = (-1)^{S_{\text{aub}}}$$

$$\begin{aligned} \odot \quad \langle e, a | m, b \rangle &= \langle e, o | M(a) E(b) | m, o \rangle \\ &= (-1)^{S_{\text{aub}}} \langle e, o | E(b) M(a) | m, o \rangle \\ &= (-1)^{S_{\text{aub}}} \langle e, o | m, o \rangle \\ &= (-1)^{S_{\text{aub}}} \end{aligned}$$

Then we find that

$$|m, a\rangle = \frac{1}{|H(\mathbb{C}, \mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_b (-1)^{S_{\text{aub}}} |e, b\rangle$$

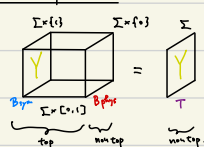
Prop

$$\langle m, a | T_0 \rangle = Z_{\text{oc}(T_0)}[a]$$

$$\begin{aligned} \odot \quad \text{LHS} &= \frac{1}{|H(\mathbb{C}, \mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_b \langle m, a | e, b \rangle \langle e, b | T_0 \rangle \\ &= \sum_b (-1)^{S_{\text{aub}}} Z_{T_0}[b] \\ &= Z_{\text{oc}(T_0)}[a] \\ &= \text{RHS} \end{aligned}$$

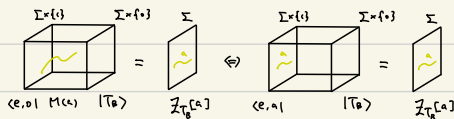
→ We can reinterpret the gauging as a change of basis,
or "baby".

SymTFT picture

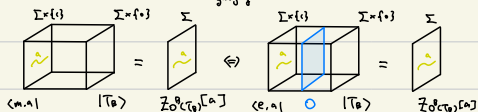


Phys: dynamical } baby
Bryn: sym op eff } baby → only capture sym.

In our setup, let us define $|T_0\rangle$ s.t. $Z_{T_0}[a] = \langle e, a | T_0 \rangle$



change of Bryn. ↓ gauging
insertion of 0 domain wall
new gauging ↓ gauging



Basis w.r.t. $F(a)$

Define $|1\rangle$ by $F(a)|1\rangle = (-1)^{Z_1(a)} |1\rangle \quad \forall a \in H(\mathbb{C}, \mathbb{Z}_2)$

$$|1+a\rangle \text{ by } |1+a\rangle := E(a)|1\rangle$$

$$\leadsto V(\Sigma) \cong \text{span}_{\mathbb{C}} \{|1+a\rangle \mid a \in H(\mathbb{C}, \mathbb{Z}_2)\}$$

Normalization

$$\langle e, o | 1 \rangle = 1$$

$$\langle 1 | 1' \rangle = |H(\mathbb{C}, \mathbb{Z}_2)|^{\frac{1}{2}} \delta_{11'} \leadsto \langle 1 | T_0 \rangle = Z_{F_{\text{oc}}(T_0)}[1]$$

$$\text{id}_{V(\Sigma)} = \frac{1}{|H(\mathbb{C}, \mathbb{Z}_2)|^{\frac{1}{2}}} \cdot \sum_1 |1\rangle \langle 1|$$

Summary

Sym part of d-din thng is described by d+1-din TQFT.

As an example, we saw Duality Web in 3+1D can be described by TC in 2+1D.

§ 4. Condensation defects with $\mathbb{Z}_2$ gauge theory

Recall

Gauging $\mathbb{Z}_2$ on T^2

$$= \frac{1}{2} \cdot (\square + \boxed{} + \boxed{} + \boxed{})$$

We insert 1D defect on 2D spacetime mfd.

→ What if we do that for codim- g defect on codim- p submfd?

$$\begin{aligned} \text{Se} \rightarrow & \frac{1}{2} \left(\begin{array}{c} e \\ | \\ | \\ | \end{array} + \begin{array}{c} e \\ | \\ \text{red loop} \\ | \end{array} + \begin{array}{c} e \\ | \\ | \\ \text{red loop} \end{array} \right) \\ &= \frac{1}{2} \left(\begin{array}{c} | \\ | \\ | \end{array} + \begin{array}{c} | \\ | \\ | \end{array} + \begin{array}{c} | \\ | \\ | \end{array} \right) \\ &= \begin{array}{c} | \\ | \\ | \end{array} + \begin{array}{c} | \\ | \\ | \end{array} \end{aligned}$$

→ Non-inv 0-form sym

Def

• Higher gauging

⊕ Summing over insertions of codim- $(p+1)$ top defect

on codim- p mfd. w/ $p \leq g-1$.

i.e. gauging a g -form sym on a codim- p mfd.

• Condensation defect

⊕ Top defect resulting from higher gauging



gauging a on Σ

Fact

∀ 0-form sym in 2+1D TQFT arises from

higher gauging of 1-form sym.

Fact

Condensation surfaces in TC have forms as

$$\text{Se} := |e, 0\rangle \langle e, 0|, \quad S_m := |m, 0\rangle \langle m, 0|, \quad S_{em} := |e, 0\rangle \langle m, 0| \text{ etc...}$$

→ Let us check how they work.

E.g.

For TC: $\{1, e, m, f\}$, we can construct surface operators

Se, S_m, S_f by higher gauging.

$$\begin{aligned} \text{Se} \rightarrow & \frac{1}{2} \left(\begin{array}{c} e \\ | \\ | \\ | \end{array} + \begin{array}{c} e \\ | \\ \text{red loop} \\ | \end{array} + \begin{array}{c} e \\ | \\ | \\ \text{red loop} \end{array} \right) \\ &= \frac{1}{2} \left(\begin{array}{c} | \\ | \\ | \end{array} - \begin{array}{c} | \\ | \\ | \end{array} + \begin{array}{c} | \\ | \\ | \end{array} + \begin{array}{c} | \\ | \\ | \end{array} \right) \\ &= \begin{array}{c} | \\ | \\ | \end{array} \\ &= \begin{array}{c} | \\ | \\ | \end{array} \end{aligned}$$

→ invertible 0-form sym

$$\begin{aligned} \text{⊗} \quad \begin{array}{c} \text{red loop} \\ | \\ | \\ | \end{array} &= - \begin{array}{c} | \\ | \\ | \end{array} \\ &= - \begin{array}{c} | \\ | \\ | \end{array} \\ &= (-1)^{\frac{1}{2}} \begin{array}{c} | \\ | \\ | \end{array} \end{aligned}$$

Recall

• $V(T^2) \cong \text{span}\{|1\rangle, |e\rangle, |m\rangle, |f\rangle\}$ w/ $\langle a|b\rangle = \delta_{ab}$

w/ $|e\rangle := E(x)|0\rangle$ etc...

• $E(x)|e, 0\rangle = |e, 0\rangle \quad \forall x \in H^1(\Sigma, \mathbb{Z}_2)$

→ How can we describe $|e, a\rangle$ w/ $\{ |a\rangle \}_{a \in \mathbb{Z}}$?

$$\hookrightarrow |e, 0\rangle = |1\rangle + |e\rangle$$

⊙ Set $|e, 0\rangle = \sum C_a |a\rangle \quad \forall C_a \in \mathbb{C}, a \in \{1, e, m, f\}$

$$E(x)|e, 0\rangle = |e, 0\rangle \Rightarrow \text{LHS} = C_1|e\rangle + C_e|1\rangle + C_m|f\rangle + C_f|m\rangle$$

$$\therefore C_1 = C_e, C_m = C_f$$

$$E(m)|e, 0\rangle = |e, 0\rangle \neq \text{LHS} = C_1|1\rangle + C_e|e\rangle - C_m|m\rangle - C_f|f\rangle$$

$$\therefore C_m = C_f = 0$$

∴ We obtain $|e, 0\rangle = |1\rangle + |e\rangle$

$$|e, 2\rangle = |m\rangle + |f\rangle$$

$$|e, 4\rangle = |1\rangle - |e\rangle$$

$$|e, 6\rangle = |m\rangle - |f\rangle$$

$$\therefore S_e |a\rangle = \begin{cases} |1\rangle + |e\rangle & a \in \{1, e\} \\ 0 & a \in \{m, f\} \end{cases}$$

→ One can compute $S_m, S_{em}, \dots$

Prop

$$S_f := \frac{1}{|H(\psi, \mathbf{a})|^{1/2}} \sum_{\gamma} c_{-1} A_f^{(\gamma)} \quad |\gamma\rangle \in \mathcal{H}_1 : \mathbb{Z}_2 \text{ o-form gm of } e \leftrightarrow m$$

⊙

$$\begin{aligned} S_f E(a) &= \frac{1}{|H(\psi, \mathbf{a})|^{1/2}} \sum_{\gamma} c_{-1} A_f^{(\gamma)} |\gamma\rangle \in \mathcal{H}_1 \\ &= \frac{1}{|H(\psi, \mathbf{a})|^{1/2}} \sum_{\gamma} c_{-1} A_f^{(\gamma+a)} |\gamma+a\rangle \in \mathcal{H}_1 \\ &= \frac{1}{|H(\psi, \mathbf{a})|^{1/2}} \sum_{\gamma} c_{-1} A_f^{(\gamma)} E(a) \cdot c_{-1} \gamma(a) |\gamma\rangle \in \mathcal{H}_1 \\ &= E(a) \cdot F(a) \frac{1}{|H(\psi, \mathbf{a})|^{1/2}} \sum_{\gamma} c_{-1} A_f^{(\gamma)} |\gamma\rangle \in \mathcal{H}_1 \\ &= M(a) S_f \end{aligned}$$