

Web of dualities on non-orientable surfaces

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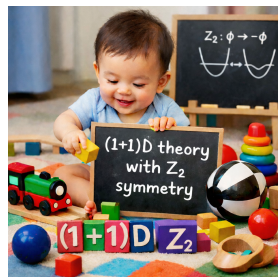
Based on [IO-Tsuji 2601.09067].

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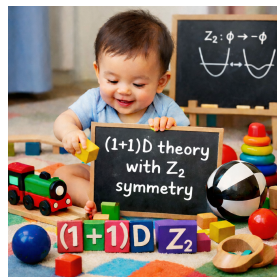


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→ Topological manipulations such as

- Gauging,
- Fermionizing,
- Stacking.



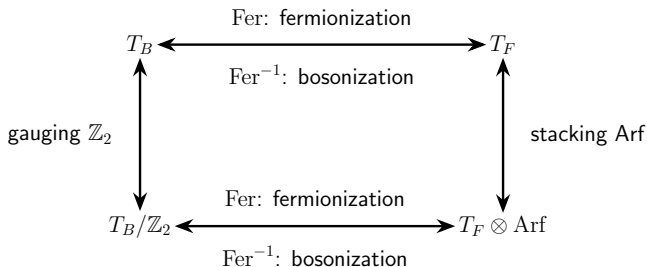
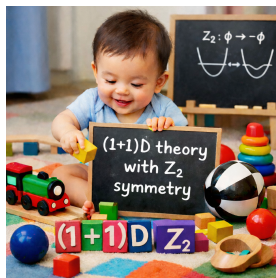
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→ Duality web [Karch et al. 1902.05550]:



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- Can we construct a non-orientable duality web?
- Yes! [Spieler 2504.20021]
- We study more about this duality web from several aspects:

- Symmetry TFT (SymTFT)
- Sectors of Hilbert space
- Group structure of topological manipulations

- Duality web on orientable surfaces
 - $\mathbb{Z}_2$ symmetry and its gauging
 - $\mathbb{Z}_2$ -quadratic refinement, Arf invariant, and spin structures
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These are familiar concepts in QFT $\rightarrow$ What changes when $G = \mathbb{Z}_2$?

Fact:

For $G = \mathbb{Z}_2$, partition functions and a path integral are given as

$$Z_{T_B}[a] \quad a \in H^1(\Sigma; \mathbb{Z}_2), \quad \int \mathcal{D}A = \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|^{1/2}} \sum_{a \in H^1(\Sigma; \mathbb{Z}_2)}$$

Therefore, $\mathbb{Z}_2$ gauging is given as

$$Z_{T_B/\mathbb{Z}_2} = \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|^{1/2}} \sum_{a \in H^1(\Sigma; \mathbb{Z}_2)} Z_{T_B}[a]$$

e.g. For $\Sigma = T^2$, with $H^1(T^2; \mathbb{Z}_2) \cong \{0, x, y, x + y\}$, we obtain

$$\begin{aligned} Z_{T_B/\mathbb{Z}_2} &= \frac{1}{2} (Z_{T_B}[0] + Z_{T_B}[x] + Z_{T_B}[y] + Z_{T_B}[x + y]) \\ &= \frac{1}{2} \left(\begin{array}{c} \text{square with arrows on all four edges} \end{array} + \begin{array}{c} \text{square with arrows on all four edges and a red horizontal line} \end{array} + \begin{array}{c} \text{square with arrows on all four edges and a red vertical line} \end{array} + \begin{array}{c} \text{square with arrows on all four edges and a red cross} \end{array} \right). \end{aligned}$$

We can consider refined version of gauging so that resulting theory $T_B/\mathbb{Z}_2$ has dual $\mathbb{Z}_2$ symmetry:

$$Z_{O^B(T_B)}[a] := \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|^{1/2}} \sum_{b \in H^1(\Sigma; \mathbb{Z}_2)} (-1)^{\int a \cup b} Z_{T_B}[b]$$

Here, $a \in H^1(\Sigma; \mathbb{Z}_2)$ is a background field of dual $\mathbb{Z}_2$ symmetry and cup product $\cup$ is a discrete version of wedge product $\wedge$.

Remark:

$(O^B)^2 = 1$, i.e. gauging twice takes you back to the original theory.

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Fermionic theory depends on spin structures of the spacetime.

e.g. We have four choices of spin structures on torus T^2 ;

$(x, y) = (\text{NS}, \text{NS}), (\text{NS}, \text{R}), (\text{R}, \text{NS}), (\text{R}, \text{R})$.

Remark: Orientable genus g surface Σ_g admits 2^{2g} spin structures.

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$$\{\text{Spin structure } \eta \text{ on } \Sigma\} \xleftrightarrow{1:1} \{q_\eta : H^1(\Sigma; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2 \text{ s.t. } \dots\}$$

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e.g. Four choices of quadratic refinement on T^2 ;

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Fact:

Spin structures are classified into two classes.

How do we classify these spin structures on T^2 ?

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Natural classification is $\{(\text{NS}, \text{NS}), (\text{NS}, \text{R}), (\text{R}, \text{NS})\}$ and $\{(\text{R}, \text{R})\}$.

Remark:

This classification is closed under $PSL(2, \mathbb{Z})$ i.e., $\tau \mapsto -1/\tau$, $\tau \mapsto \tau + 1$.

Is there any way to classify them in general? $\rightarrow$ **Arf invariant**

Arf invariant

$$Z_{\text{Arf}}[\eta] := \exp(\pi i \text{Arf}(\eta)) = \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|^{1/2}} \sum_{a \in H^1(\Sigma; \mathbb{Z}_2)} \exp(\pi i q_\eta(a))$$

e.g. Four choices of spin structures on T^2 ;

$$Z_{\text{Arf}}[(\text{NS}, \text{NS})] = \frac{1}{4^{1/2}} (1 + 1 + 1 - 1) = 1$$

$$Z_{\text{Arf}}[(\text{NS}, \text{R})] = \frac{1}{4^{1/2}} (1 + 1 - 1 + 1) = 1$$

$$Z_{\text{Arf}}[(\text{R}, \text{NS})] = \frac{1}{4^{1/2}} (1 - 1 + 1 + 1) = 1$$

$$Z_{\text{Arf}}[(\text{R}, \text{R})] = \frac{1}{4^{1/2}} (1 - 1 - 1 - 1) = -1$$

→ Arf invariant classifies $\{(\text{NS}, \text{NS}), (\text{NS}, \text{R}), (\text{R}, \text{NS})\}$ and $\{(\text{R}, \text{R})\}$.

Physically Arf invariant is a partition function of the Kitaev chain.

Mathematically Arf invariant gives a homomorphism $\Omega_2^{\text{Spin}}(\text{pt}) \xrightarrow{\sim} \mathbb{Z}_2$.

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How do we construct a fermionic theory, i.e. a theory which depends on a spin structure η .

What we have now are

- Bosonic theory with $\mathbb{Z}_2$ symmetry $Z_{TB}[a]$ where $a \in H^1(\Sigma; \mathbb{Z}_2)$
- Quadratic refinement $q_\eta : H^1(\Sigma; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ which has a one-to-one correspondence with a spin structure η

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→ Fermionization $\text{Fer} : \{\text{bosonic theory}\} \rightarrow \{\text{fermionic theory}\}$ is given as

$$Z_{\text{Fer}(T_B)}[\eta] := \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|^{1/2}} \sum_{a \in H^1(\Sigma; \mathbb{Z}_2)} \exp(\pi i q_\eta(a)) Z_{T_B}[a]$$

→ Bosonization $\text{Fer}^{-1} : \{\text{fermionic theory}\} \rightarrow \{\text{bosonic theory}\}$ is given as

$$Z_{\text{Fer}^{-1}(T_F)}[a] := \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|^{1/2}} \sum_{\eta \in \text{Spin}(\Sigma)} \exp(\pi i q_\eta(a)) Z_{T_F}[\eta]$$

What corresponds to gauging O^B in fermionic theories?

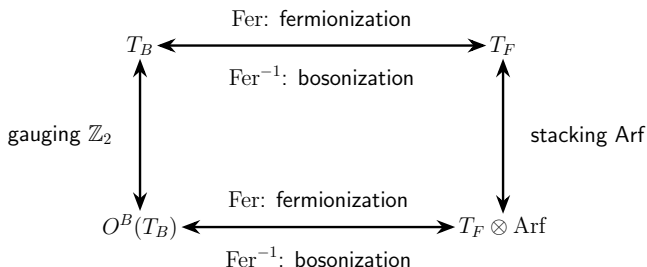
i.e. $O^F := \text{Fer} \circ O^B \circ \text{Fer}^{-1} : \{\text{fermionic theory}\} \rightarrow \{\text{fermionic theory}\}$

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One can check that

$$Z_{\text{Fer} \circ O^B \circ \text{Fer}^{-1}(T_F)}[\eta] = Z_{\text{Arf}}[\eta] Z_{T_F}[\eta].$$

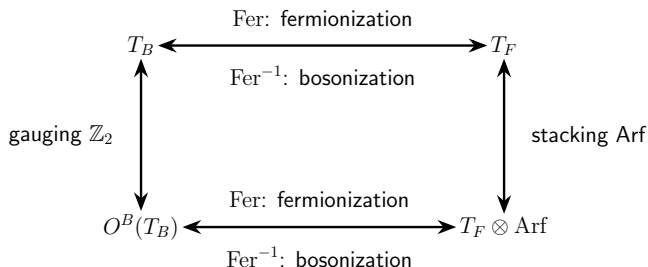


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Remark: $Z_{\text{Arf}}[\eta] \in \mathbb{Z}_2 \Leftrightarrow (O^B)^2 = 1$

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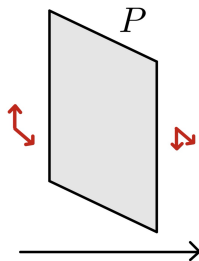
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In $(d + 1)$ D theories, what does it mean that spacetime is non-orientable?

→ **Existence of the orientation reversing defect** $P \in H_d(M^{d+1}; \mathbb{Z}_2)$.

Using Poincaré duality, we can
define [Thorngren 1404.4385]

$$w_1 := \text{PD}(P) \in H^1(M^{d+1}; \mathbb{Z}_2).$$



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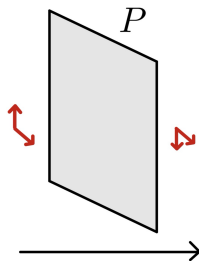
Recall:

In $(3 + 1)$ D Maxwell theory, we have

$$\int_{\Sigma} \frac{F}{2\pi} \in \mathbb{Z}.$$

If $\int_{\Sigma} \frac{F}{2\pi} \neq 0$, we cannot define $U(1)$ gauge field globally.

→ $c_1 := \frac{F}{2\pi} \in H^2(M^{3+1}; \mathbb{Z})$ is the **obstruction** to the existence of the globally-defined gauge field, i.e. to the triviality of the bundle.



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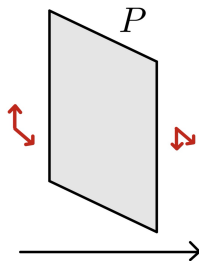
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Remark: w_1 is the obstruction to orientability.

In the presence of nontrivial w_1 , a fermionic theory depends on **Pin structure**, i.e., non-orientable analogue of a spin structure.

Recall:

On an orientable surface, we take $R(\theta) \in SO(2)$, and let $\tilde{R}(\theta) \in \text{Spin}(2)$ be its lift. Then

$$R(2\pi) = 1, \quad \tilde{R}(2\pi) = (-1)^F.$$

Thus

$$\tilde{R}(2\pi)\psi = \begin{cases} \psi & \text{if } \psi \text{ is bosonic,} \\ -\psi & \text{if } \psi \text{ is fermionic.} \end{cases}$$

What changes on non-orientable surfaces?

On a non-orientable surface, reflections $P \in O(2)$ are allowed $P^2 = 1$. However a lift $\tilde{P}$ can satisfy either

$$\tilde{P}^2 = 1 \text{ or } \tilde{P}^2 = (-1)^F.$$

These two possibilities define the two double covers Pin^+ and Pin^- . In this talk we use Pin^- , so the lifted reflection satisfies

$$\tilde{P}^2 = (-1)^F.$$

Equivalently,

$$\tilde{P}^2 \psi = \begin{cases} \psi & \text{if } \psi \text{ is bosonic,} \\ -\psi & \text{if } \psi \text{ is fermionic.} \end{cases}$$

Hereafter, we write $\tilde{P} \in \text{Pin}^-(2)$ simply as P .

Fact:

- $\{\text{Pin}^- \text{ structure } \eta \text{ on } \Sigma\} \xleftrightarrow{1:1} \{q_\eta : H^1(\Sigma; \mathbb{Z}_2) \rightarrow \mathbb{Z}_4 \text{ s.t. } \dots\}$
- Pin^- structures are classified into eight classes.
- ABK invariant valued in $\mathbb{Z}_8$ classifies them.

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→ Stacking with SPT phases! $(-1)^{\int a \cup w_1}, (-1)^{\int w_1 \cup w_1}$.

- SPT (Symmetry-Protected Topological) phase: a TQFT whose Hilbert space on any closed spatial manifold is one-dimensional.
- Stacking: At the level of partition functions, stacking corresponds to multiplication.

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→ We can define operations S_1^B, S_2^B as follows:

$$Z_{S_1^B(T_B)}[a] := (-1)^{\int a \cup w_1} Z_{T_B}[a]$$

$$Z_{S_2^B(T_B)}[a] := (-1)^{\int w_1 \cup w_1} Z_{T_B}[a]$$

	orientable	non-orientable
w_1	trivial	nontrivial
str	Spin	Pin^-
quad ref	$q_\eta : H^1(\Sigma; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$	$q_\eta : H^1(\Sigma; \mathbb{Z}_2) \rightarrow \mathbb{Z}_4$
Inv	$\text{Arf}(\eta) \propto \sum_a e^{\pi i q_\eta(a)} \in \mathbb{Z}_2$	$\text{ABK}(\eta) \propto \sum_a e^{\frac{\pi i}{2} q_\eta(a)} \in \mathbb{Z}_8$
Fer	$Z_{\text{Fer}(T_B)} \propto \sum_a e^{\pi i q_\eta(a)} Z_{T_B}[a]$	$Z_{\text{Fer}(T_B)} \propto \sum_a e^{\frac{\pi i}{2} q_\eta(a)} Z_{T_B}[a]$
SPT	trivial	$(-1)^{\int a \cup w_1}, (-1)^{\int w_1 \cup w_1}$

Table: Summary of orientable vs non-orientable

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Recall:

- Fourier transform of a wave function in Quantum Mechanics

$$\mathcal{F}(\psi)(p) := \tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int dx e^{-ipx} \psi(x).$$

- Gauging $\mathbb{Z}_2$ in our set-up

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$$\mathcal{F}(\psi)(p) := \tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int dx e^{-ipx} \psi(x).$$

- Gauging $\mathbb{Z}_2$ in our set-up

$$Z_{OB(T_B)}[a] = \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|^{1/2}} \sum_b (-1)^{\int a \cup b} Z_{T_B}[b].$$

Observation:

Gauging can be interpreted as a discrete Fourier transform.

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Observation:

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Remark:

$\mathcal{F}$ is a $\mathbb{Z}_2$ operation $\mathcal{F} \circ \mathcal{F}(\psi)(x) = \psi(-x) = \psi(x)$ for even functions.
 $\rightarrow$ Compatible with $(O^B)^2 = 1$.

Recall:

We can interpret the Fourier transform as a change of basis:

$$\mathcal{F}(\psi)(p) = \langle p | \psi \rangle = \langle p | \int dx | x \rangle \langle x | \psi \rangle = \frac{1}{\sqrt{2\pi}} \int dx e^{-ipx} \psi(x)$$

where $\langle p | x \rangle = \frac{1}{\sqrt{2\pi}} e^{-ipx}$.

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Fact:

Gauging can be expressed as

$$\begin{aligned} Z_{OB(T_B)}[a] &= \langle m, a | T_B \rangle = \langle m, a | \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|^{1/2}} \sum_b |e, b \rangle \langle e, b | T_B \rangle \\ &= \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|^{1/2}} \sum_b (-1)^{\int a \cup b} Z_{T_B}[b] \end{aligned}$$

where $\{|e, a\rangle\}$ and $\{|m, a\rangle\}$ are boundary conditions in $(2+1)\text{D}$ TQFT,
i.e. **gauging in $(1+1)\text{D}$ can be described by $(2+1)\text{D}$ TQFT.**

SymTFT is a framework to understand symmetries in $(d+1)$ D theories in terms of $(d+2)$ D TQFTs.

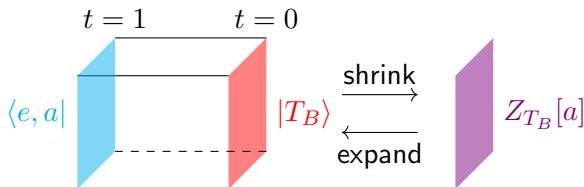


Figure: $|T_B\rangle/|e, a\rangle$ correspond to a physical/gapped boundary respectively, where the spacetime is given as $\Sigma \times [0, 1]$.

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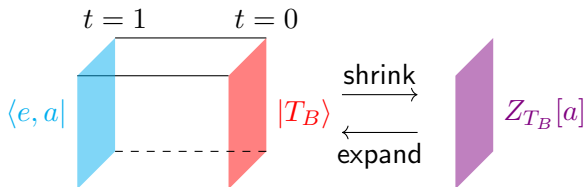


Figure: $|T_B\rangle/|e, a\rangle$ correspond to a physical/gapped boundary respectively, where the spacetime is given as $\Sigma \times [0, 1]$.

Fact:

SymTFT for $\mathbb{Z}_2$ symmetry is a Toric code (i.e. $\mathbb{Z}_2$ gauge theory).

Toric code has line operators $\{1, E(a), M(a), F(a) := E(a)M(a)\}$ s.t.

$$E(a)E(b) = E(a + b), \quad M(a)M(b) = M(a + b).$$

$$E(a)M(b) = (-1)^{\int a \cup b} M(b)E(a), \quad a, b \in H^1(\Sigma; \mathbb{Z}_2).$$

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→ Define the gapped boundary condition $|e, 0\rangle$ by condensing $E(a)$:

$$E(a)|e, 0\rangle = |e, 0\rangle \quad \text{for all } a \in H^1(\Sigma; \mathbb{Z}_2)$$

Then one can check that $|e, b\rangle := M(b)|e, 0\rangle$ is an eigenvector of $E(a)$:

$$E(a)M(b)|e, 0\rangle = (-1)^{\int a \cup b} M(b)E(a)|e, 0\rangle = (-1)^{\int a \cup b} M(b)|e, 0\rangle$$

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$$E(a)M(b)|e, 0\rangle = (-1)^{\int a \cup b} M(b)E(a)|e, 0\rangle = (-1)^{\int a \cup b} M(b)|e, 0\rangle$$

→ The Hilbert space on Σ assigned by $(2+1)\text{D}$ TQFT is given as

$$\mathcal{H}(\Sigma) = \text{span}_{\mathbb{C}}\{|e, a\rangle \mid a \in H^1(\Sigma; \mathbb{Z}_2)\}$$

We can define another boundary condition by condensing $M(a)$:

$$M(a)|m, 0\rangle = |m, 0\rangle \quad \text{for all } a \in H^1(\Sigma; \mathbb{Z}_2)$$

Then we obtain another basis:

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One can check that these bases are related as

$$|m, a\rangle = \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|^{1/2}} \sum_b (-1)^{f^{a \cup b}} |e, b\rangle$$

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Now we define $|T_B\rangle$ so that $\langle e, a|T_B\rangle = Z_{T_B}[a]$. Then we obtain

$$Z_{O^B(T_B)}[a] = \langle m, a|T_B\rangle = \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|^{1/2}} \sum_b (-1)^{\int a \cup b} Z_{T_B}[b]$$

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→ We construct SymTFT for O^B, S_1^B, S_2^B , Fer on non-orientable surfaces.

- Duality web on orientable surfaces
 - $\mathbb{Z}_2$ symmetry and its gauging
 - $\mathbb{Z}_2$ -quadratic refinement, Arf invariant, and spin structures
 - Fermionization and Bosonization
 - What changes on non-orientable surfaces
- Duality web on non-orientable surfaces
 - SymTFT
 - Sectors of Hilbert space
 - Group structure of topological manipulations

We can study the duality from the perspectives of the Hilbert space!

Let us first study the case of orientable case. e.g. [Hsieh et al. 2002.12283]

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The Hilbert space on S^1 can be decomposed as

$$\mathcal{H}_{\text{all}} = \mathcal{H} \oplus \mathcal{H}^g \quad \text{with} \quad \begin{cases} \mathcal{H} = \mathcal{H}_{\text{even}} \oplus \mathcal{H}_{\text{odd}} \\ \mathcal{H}^g = \mathcal{H}_{\text{even}}^g \oplus \mathcal{H}_{\text{odd}}^g \end{cases}$$

Here subscripts even/odd means the eigenvalues of $g : \mathcal{H}_{\text{all}} \rightarrow \mathcal{H}_{\text{all}}$:

$$g\psi = \begin{cases} \psi & \psi \in \mathcal{H}_{\text{even}} \oplus \mathcal{H}_{\text{even}}^g \\ -\psi & \psi \in \mathcal{H}_{\text{odd}} \oplus \mathcal{H}_{\text{odd}}^g \end{cases}$$

and superscripts mean untwisted/twisted by g .

T_B	untwisted : $\mathcal{H}$	twisted $\mathcal{H}^g$
even	S	U
odd	T	V

Table: Sectors of T_B on the circle S^1 .

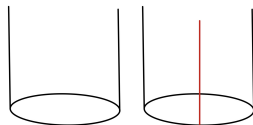


Figure: Untwisted/twisted Hilb sp.

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$$Z_{T_B}[T^2; 0] = \text{Tr}_{\mathcal{H}}^{T_B}(e^{-lH}) = \text{diagram 1}$$


$$Z_{T_B}[T^2; x] = \text{Tr}_{\mathcal{H}}^{T_B}(ge^{-lH}) = \text{diagram 2}$$


$$Z_{T_B}[T^2; y] = \text{Tr}_{\mathcal{H}^g}^{T_B}(e^{-lH}) = \text{diagram 3}$$


$$Z_{T_B}[T^2; x + y] = \text{Tr}_{\mathcal{H}^g}^{T_B}(ge^{-lH}) = \text{diagram 4} .$$


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Then, the **even**, **untwisted** sector of T_B S can be computed as

$$\text{Tr}_{\mathcal{H}}^{T_B} \left(\frac{1+g}{2} e^{-lH} \right) = \frac{1}{2} \left(\begin{array}{c} \text{---} \rightarrow \\ \uparrow \quad \downarrow \\ \text{---} \end{array} + \begin{array}{c} \text{---} \rightarrow \\ \uparrow \text{---} \downarrow \\ \text{---} \end{array} \right)_{T_B}$$

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$$Z_{T_B}[T^2; x] = \text{Tr}_{\mathcal{H}}^{T_B}(ge^{-lH}) = \begin{array}{c} \text{---} \rightarrow \\ \text{---} \text{ (red)} \text{---} \\ \uparrow \quad \downarrow \\ \text{---} \end{array}$$

$$Z_{T_B}[T^2; y] = \text{Tr}_{\mathcal{H}^g}^{T_B}(e^{-lH}) = \begin{array}{c} \text{---} \rightarrow \\ \uparrow \quad \downarrow \\ \text{---} \text{ (red)} \text{---} \\ \uparrow \quad \downarrow \\ \text{---} \end{array}$$

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$$\text{Tr}_{\mathcal{H}}^{T_B} \left(\frac{1 + g}{2} e^{-lH} \right) = \frac{1}{2} \left(\begin{array}{c} \text{---} \rightarrow \\ \uparrow \quad \downarrow \\ \text{---} \end{array} + \begin{array}{c} \text{---} \rightarrow \\ \text{---} \text{ (red)} \text{---} \\ \uparrow \quad \downarrow \\ \text{---} \end{array} \right)_{T_B}$$

→ We can keep track of these sectors in the web of dualities!

For example, we can calculate the **even**, **twisted** sector of $O^B(T_B)$ as

$$\begin{aligned}
 & \text{Tr}_{\mathcal{H}^g}^{O^B(T_B)} \left(\frac{1+g}{2} e^{-lH} \right) \\
 &= \frac{1}{2} \left(\begin{array}{|c|} \hline \text{Diagram 1} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{Diagram 2} \\ \hline \end{array} \right)_{O^B(T_B)} \\
 &= \frac{1}{2} \left(\begin{array}{|c|} \hline \text{Diagram 3} \\ \hline \end{array} - \begin{array}{|c|} \hline \text{Diagram 4} \\ \hline \end{array} \right)_{T_B} \\
 &= \text{Tr}_{\mathcal{H}}^{T_B} \left(\frac{1-g}{2} e^{-lH} \right)
 \end{aligned}$$

The diagrams are as follows:

- Diagram 1: A square with arrows on all four sides pointing clockwise. A vertical red line is drawn in the center, with arrows pointing upwards.
- Diagram 2: A square with arrows on all four sides pointing clockwise. A horizontal red line is drawn in the center, with arrows pointing to the right.
- Diagram 3: A square with arrows on all four sides pointing clockwise. A vertical red line is drawn in the center, with arrows pointing downwards.
- Diagram 4: A square with arrows on all four sides pointing clockwise. A horizontal red line is drawn in the center, with arrows pointing to the left.

which turns out to be the **odd**, **untwisted** sector of T_B .

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 &= \frac{1}{2} \left(\begin{array}{|c|} \hline \uparrow \downarrow \\ \hline \end{array} - \begin{array}{|c|} \hline \uparrow \downarrow \\ \hline \end{array} \right)_{T_B} \\
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T_B	untwisted	twisted
even	S	U
odd	T	V

$O^B(T_B)$	untwisted	twisted
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$$\begin{aligned}
 & \text{Tr}_{\mathcal{H}^g}^{O^B(T_B)} \left(\frac{1+g}{2} e^{-lH} \right) \\
 &= \frac{1}{2} \left(\begin{array}{|c|} \hline \text{square with vertical red line and arrows} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{square with horizontal red line and arrows} \\ \hline \end{array} \right)_{O^B(T_B)} \\
 &= \frac{1}{2} \left(\begin{array}{|c|} \hline \text{square with clockwise arrows} \\ \hline \end{array} - \begin{array}{|c|} \hline \text{square with counter-clockwise arrows} \\ \hline \end{array} \right)_{T_B} \\
 &= \text{Tr}_{\mathcal{H}}^{T_B} \left(\frac{1-g}{2} e^{-lH} \right)
 \end{aligned}$$

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T_B	untwisted	twisted
even	S	U
odd	T	V

$O^B(T_B)$	untwisted	twisted
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→ We can do the same for Fermionization.

Now let us try to refine this viewpoint in terms of P .

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$$\mathcal{H}_{\text{all}} = \mathcal{H} \oplus \mathcal{H}^g \quad \text{with} \quad \begin{cases} \mathcal{H} = \mathcal{H}_{\text{even},+} \oplus \mathcal{H}_{\text{even},-} \oplus \mathcal{H}_{\text{odd},+} \oplus \mathcal{H}_{\text{odd},-} \\ \mathcal{H}^g = \mathcal{H}_{\text{even},+}^g \oplus \mathcal{H}_{\text{even},-}^g \oplus \mathcal{H}_{\text{odd},+}^g \oplus \mathcal{H}_{\text{odd},-}^g \end{cases}$$

Here subscripts $+/-$ mean the eigenvalues of $P : \mathcal{H}_{\text{all}} \rightarrow \mathcal{H}_{\text{all}}$:

$$P\psi = \begin{cases} \psi & \psi \in \mathcal{H}_+ \\ -\psi & \psi \in \mathcal{H}_- \end{cases}$$

T_B		untwisted	twisted
even	$P = +1$	S_+	U_+
	$P = -1$	S_-	U_-
odd	$P = +1$	T_+	V_+
	$P = -1$	T_-	V_-

Table: Sectors of T_B on the circle S^1 .

Suppose we have a torus partition function $Z_{T_B}[T^2; 0] = \text{Tr}_{\mathcal{H}}^{T_B}(e^{-lH})$

A partition function with an insertion of $P : \mathcal{H}_{\text{all}} \rightarrow \mathcal{H}_{\text{all}}$

$$Z_{T_B}[K; 0] = \text{Tr}_{\mathcal{H}}^{T_B}(Pe^{-lH}) = \img alt="A square diagram representing a trace. The square has four arrows: the top arrow points left, the bottom arrow points right, the left arrow points up, and the right arrow points down. This configuration represents a trace operation." data-bbox="658 291 723 382"/>$$

can be interpreted as a **Klein bottle** partition function.

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A partition function with an insertion of $P : \mathcal{H}_{\text{all}} \rightarrow \mathcal{H}_{\text{all}}$

$$Z_{T_B}[K; 0] = \text{Tr}_{\mathcal{H}}^{T_B}(Pe^{-lH}) = \text{Klein bottle diagram}$$

can be interpreted as a **Klein bottle** partition function.

In terms of $g : \mathcal{H}_{\text{all}} \rightarrow \mathcal{H}_{\text{all}}$ where $g \in \mathbb{Z}_2$, we obtain

$Z_{T_B}[T^2; 0] = \text{Tr}_{\mathcal{H}}^{T_B}(e^{-lH}) =$		$Z_{T_B}[K; 0] = \text{Tr}_{\mathcal{H}}^{T_B}(Pe^{-lH}) =$	
$Z_{T_B}[T^2; x] = \text{Tr}_{\mathcal{H}}^{T_B}(g e^{-lH}) =$		$Z_{T_B}[K; x] = \text{Tr}_{\mathcal{H}}^{T_B}(gPe^{-lH}) =$	
$Z_{T_B}[T^2; y] = \text{Tr}_{\mathcal{H}^g}^{T_B}(e^{-lH}) =$		$Z_{T_B}[K; y] = \text{Tr}_{\mathcal{H}^g}^{T_B}(Pe^{-lH}) =$	
$Z_{T_B}[T^2; x + y] = \text{Tr}_{\mathcal{H}^g}^{T_B}(g e^{-lH}) =$		$Z_{T_B}[K; x + y] = \text{Tr}_{\mathcal{H}^g}^{T_B}(gPe^{-lH}) =$	

Then, the **even**, **+**, **untwisted** sector of $T_B \mathcal{U}_+$ can be computed as

$$\mathrm{Tr}_{\mathcal{H}}^{T_B} \left(\frac{1+g}{2} \frac{1+P}{2} e^{-lH} \right) = \frac{1}{4} \left(\begin{array}{c} \text{square with clockwise arrows} \\ + \text{square with counter-clockwise arrows and red top edge} \\ + \text{square with clockwise arrows} \\ + \text{square with counter-clockwise arrows and red top edge} \end{array} \right)_{T_B}$$

Then, the **even**, **+**, **untwisted** sector of T_B U_+ can be computed as

$$\text{Tr}_{\mathcal{H}}^{T_B} \left(\frac{1+g}{2} \frac{1+P}{2} e^{-lH} \right) = \frac{1}{4} \left(\begin{array}{c} \text{square with clockwise arrows} \\ + \text{square with counter-clockwise arrows and red top edge} \\ + \text{square with clockwise arrows} \\ + \text{square with counter-clockwise arrows and red top edge} \end{array} \right)_{T_B}$$

→ We can keep track of these sectors in the web of dualities!

For example, we can calculate the **odd**, **+**, **twisted** sector of $O^B(T_B)$ as

$$\begin{aligned} & \text{Tr}_{\mathcal{H}^g}^{O^B(T_B)} \left(\frac{1-g}{2} \frac{1+P}{2} e^{-lH} \right) \\ &= \frac{1}{4} \left(\begin{array}{c} \text{square with clockwise arrows and red vertical edge} \\ - \text{square with counter-clockwise arrows and red horizontal edge} \\ + \text{square with clockwise arrows and red vertical edge} \\ - \text{square with counter-clockwise arrows and red horizontal edge} \end{array} \right)_{O^B(T_B)} \\ &= \frac{1}{4} \left(\begin{array}{c} \text{square with clockwise arrows and red vertical edge} \\ - \text{square with counter-clockwise arrows and red horizontal edge} \\ - \text{square with clockwise arrows and red vertical edge} \\ + \text{square with counter-clockwise arrows and red horizontal edge} \end{array} \right)_{T_B} \\ &= \text{Tr}_{\mathcal{H}^g}^{T_B} \left(\frac{1-g}{2} \frac{1-P}{2} e^{-lH} \right) \end{aligned}$$

which turns out to be the **odd**, **-**, **twisted** sector of T_B V_- .

In this way, we give refined correspondence between T_B and $O^B(T_B)$ in terms of P eigenvalues:

T_B	$\mathcal{H}$	$\mathcal{H}^g$
even	S	U
odd	T	V



	T_B	$\mathcal{H}$	$\mathcal{H}^g$
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odd	U	V



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	$P = -1$	U_-	V_+

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even	$P = +1$	S_+	U_+
	$P = -1$	S_-	U_-
odd	$P = +1$	T_+	V_+
	$P = -1$	T_-	V_-

$O^B(T_B)$	$\mathcal{H}$	$\mathcal{H}^g$
even	S	T
odd	U	V



$O^B(T_B)$		$\mathcal{H}$	$\mathcal{H}^g$
even	$P = +1$	S_+	T_+
	$P = -1$	S_-	T_-
odd	$P = +1$	U_+	V_-
	$P = -1$	U_-	V_+

→ How about fermionization?

In fermionic theories, we have $P^2 = (-1)^F$, i.e., $P^4 = 1$. Therefore, $\mathcal{H}_{\text{all}}$ can be decomposed in terms of $P = \pm 1, \pm i$, together with NS, R:

$$\mathcal{H}_{\text{all}} = \mathcal{H}_{\text{NS}} \oplus \mathcal{H}_{\text{R}} \quad \text{with} \quad \begin{cases} \mathcal{H}_{\text{NS}} = \mathcal{H}_{\text{NS},+1} \oplus \mathcal{H}_{\text{NS},+i} \oplus \mathcal{H}_{\text{NS},-1} \oplus \mathcal{H}_{\text{NS},-i}, \\ \mathcal{H}_{\text{R}} = \mathcal{H}_{\text{R},+1} \oplus \mathcal{H}_{\text{R},+i} \oplus \mathcal{H}_{\text{R},-1} \oplus \mathcal{H}_{\text{R},-i}. \end{cases}$$

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For example, the $\text{R}, +i$ sector of $\text{Fer}(T_B)$ can be computed as

$$\begin{aligned} \text{Tr}_{\mathcal{H}_{\text{R}}}^{\text{Fer}(T_B)} \left(\frac{1 - iP - P^2 + iP^3}{4} e^{-lH} \right) \\ = \frac{1}{4} \left(\begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \\ \text{diagram 3} \\ \text{diagram 4} \end{array} \right)_{\text{Fer}(T_B)} \\ = \frac{1}{4} \left(\begin{array}{c} \text{diagram 5} \\ \text{diagram 6} \\ \text{diagram 7} \\ \text{diagram 8} \end{array} \right)_{T_B} \end{aligned}$$

The diagrams are square loops with arrows on the edges. Red lines indicate specific configurations: a vertical red line, a horizontal red line, or a cross (both vertical and horizontal red lines).

which turns out to be the $\text{even}, -, \text{twisted}$ sector U_- .

With the same procedure, we obtain the following table:

T_B		$\mathcal{H}$	$\mathcal{H}^g$
even	$P = +1$	S_+	U_+
	$P = -1$	S_-	U_-
odd	$P = +1$	T_+	V_+
	$P = -1$	T_-	V_-

$O^B(T_B)$		$\mathcal{H}$	$\mathcal{H}^g$
even	$P = +1$	S_+	T_+
	$P = -1$	S_-	T_-
odd	$P = +1$	U_+	V_-
	$P = -1$	U_-	V_+

$S_1^B(T_B)$		$\mathcal{H}$	$\mathcal{H}^g$
even	$P = +1$	S_+	U_-
	$P = -1$	S_-	U_+
odd	$P = +1$	T_+	V_-
	$P = -1$	T_-	V_+

T_F	NS	R
$P = +1$	S_+	T_+
$P = -1$	S_-	T_-
$P = +i$	V_+	U_-
$P = -i$	V_-	U_+

$O^F(T_F)$	NS	R
$P = +1$	S_+	U_+
$P = -1$	S_-	U_-
$P = +i$	V_-	T_-
$P = -i$	V_+	T_+

$S_1^F(T_F)$	NS	R
$P = +1$	S_+	T_+
$P = -1$	S_-	T_-
$P = +i$	V_-	U_+
$P = -i$	V_+	U_-

Table: Summary of the dualities among sectors.

We refined the known dualities among S, T, U, V in terms of P !

- Duality web on orientable surfaces
 - $\mathbb{Z}_2$ symmetry and its gauging
 - $\mathbb{Z}_2$ -quadratic refinement, Arf invariant, and spin structures
 - Fermionization and Bosonization
 - What changes on non-orientable surfaces
- Duality web on non-orientable surfaces
 - SymTFT
 - Sectors of Hilbert space
 - Group structure of topological manipulations

Recall:

We have topological manipulations on bosonic theories as follows:

$$Z_{O^B(T_B)}[a] = \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|^{1/2}} \sum_{b \in H^1(\Sigma; \mathbb{Z}_2)} (-1)^{\int a \cup b} Z_{T_B}[b]$$

$$Z_{S_1^B(T_B)}[a] = (-1)^{\int a \cup w_1} Z_{T_B}[a]$$

$$Z_{S_2^B(T_B)}[a] = (-1)^{\int w_1 \cup w_1} Z_{T_B}[a]$$

Recall:

We have topological manipulations on bosonic theories as follows:

$$Z_{O^B(T_B)}[a] = \frac{1}{|H^1(\Sigma; \mathbb{Z}_2)|^{1/2}} \sum_{b \in H^1(\Sigma; \mathbb{Z}_2)} (-1)^{\int a \cup b} Z_{T_B}[b]$$

$$Z_{S_1^B(T_B)}[a] = (-1)^{\int a \cup w_1} Z_{T_B}[a]$$

$$Z_{S_2^B(T_B)}[a] = (-1)^{\int w_1 \cup w_1} Z_{T_B}[a]$$

One can check that

$$(O^B)^2 = (S_1^B)^2 = (S_2^B)^2 = 1, (O^B S_1^B)^4 = S_2^B$$

Thus, the independent relations are as follows:

$$(O^B)^2 = (S_1^B)^2 = 1, (O^B S_1^B)^8 = 1$$

→ The dihedral group D_8 , i.e., $D_8 = \langle s, t \mid s^2 = t^2 = 1, (st)^8 = 1 \rangle$

We have 16 elements in D_8 given as

$$(O^B S_1^B)^n, S_1^B (O^B S_1^B)^n \quad n = 0, 1, \dots, 7$$

→ We can easily see the D_8 structure on fermionic theories:

$$(O^F S_1^F)^n, S_1^F (O^F S_1^F)^n \quad n = 0, 1, \dots, 7$$

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→ We can easily see the D_8 structure on fermionic theories:

$$(O^F S_1^F)^n, S_1^F (O^F S_1^F)^n \quad n = 0, 1, \dots, 7$$

As a result, we obtain the following web:

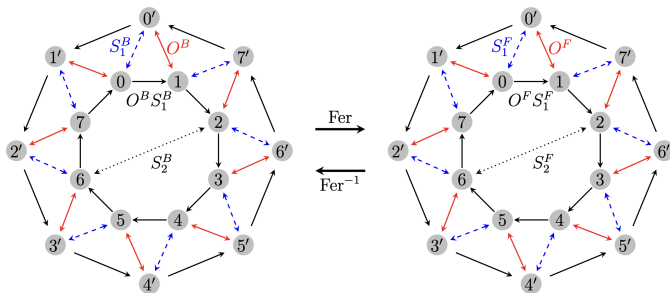


Figure: O^B : gauging $\mathbb{Z}_2$, S_1^B : stacking $(-1)^{f a \cup w_1}$, S_2^B : stacking $(-1)^{f w_1 \cup w_1}$

Results: Investigating non-orientable web of dualities in terms of

- SymTFT,
- Sectors of Hilbert space,
- Group structure of topological manipulations.

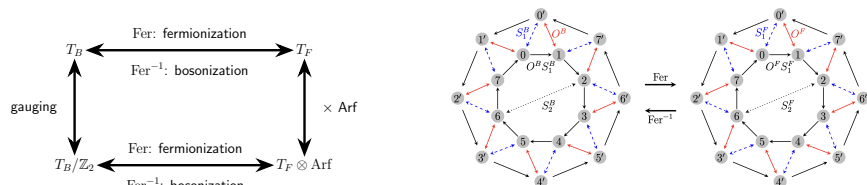


Figure: Orientable/non-orientable duality web

Future work:

- Extending to $\mathbb{Z}_n$ gauging and parafermionization.
- Studying the self duality of Ising/Majorana CFTs, etc...

Thank you!