

Topological Quantum Field Theory with time-reversal symmetry

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Can we give a rigorous construction of a
Topological Quantum Field Theory on
non-orientable manifolds?

In this talk, we focus on $(2 + 1)$ -dimensional bosonic TQFTs.

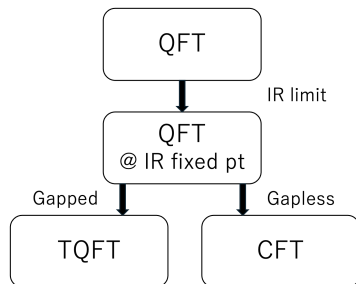
- What is TQFT
- What is TQFT with time-reversal symmetry
- Reshetikhin–Turaev TQFT
- Toward formulation of non-orientable TQFT
- Insights from crosscap

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What is TQFT

A **Topological Quantum Field Theory (TQFT)** is

- a QFT defined mathematically,
- independent of the spacetime metric,
- used to describe physics at IR fixed point.



A TQFT is a symmetric monoidal functor:

$$Z : \text{Bord}_{2,3}(\mathcal{C}) \rightarrow \text{Vect}_{\mathbb{C}},$$

where $\mathcal{C}$ is a modular tensor category (MTC) encoding anyon data.

What is TQFT

Example: $U(1)$ Chern–Simons theory

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int_M A \wedge dA$$

where $k \in \mathbb{Z}$ is a level and A is a $U(1)$ gauge field.

→ $\frac{\delta S_{\text{CS}}}{\delta g_{\mu\nu}} = 0$: independent of metric.

→ $Z[M] := \int \mathcal{D}A \exp(iS_{\text{CS}})$ depends only on M (and k).

Physical observable: Wilson loop:

$$W(\gamma, a) = \exp \left(2\pi i a / k \int_{\gamma} A \right)$$

where γ is a loop in M and $a \in \mathbb{Z}/k\mathbb{Z} \cong \{0, 1, \dots, k-1\}$

→ $\langle W(\gamma, a) \rangle$ is computed as a knot invariant.

What is TQFT

Example: $U(1)$ Chern–Simons theory

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int_M A \wedge dA$$

where $k \in \mathbb{Z}$ is a level and A is a $U(1)$ gauge field.

Input data: $k \in \mathbb{Z}$

Wilson loops: $a, b, \dots \in \mathbb{Z}/k\mathbb{Z}$

Fusions: $a + b \in \mathbb{Z}/k\mathbb{Z}$

Topological spins: $\theta(a) := \exp(\pi i a^2 / k)$

Braidings: $B(a, b) := \exp(2\pi i ab / k)$

What is TQFT

Example: $U(1)^N$ Chern–Simons theory

$$S_{\text{CS}}[A] = \frac{1}{4\pi} \int_M K_{IJ} A^I \wedge dA^J$$

where $K = (K_{IJ}) \in \text{Mat}_N(\mathbb{Z})$ is a K matrix and $\{A^I\}_{I=1,\dots,N}$ are $U(1)$ gauge fields.

Input data: $k \in \mathbb{Z} \rightarrow K \in \text{Mat}_N(\mathbb{Z})$

Wilson loops: $a, b, \dots \in \mathbb{Z}/k\mathbb{Z} \rightarrow a, b, \dots \in \mathbb{Z}^N / K\mathbb{Z}^N$

Fusions: $a + b \in \mathbb{Z}/k\mathbb{Z} \rightarrow a + b \in \mathbb{Z}^N / K\mathbb{Z}^N$

Topological spins: $\theta(a) := \exp(\pi i a^2 / k) \rightarrow \theta(a) := \exp(\pi i a^T K^{-1} a)$

Braidings: $B(a, b) := \exp(2\pi i ab / k) \rightarrow B(a, b) := \exp(2\pi i a^T K^{-1} b)$

What is TQFT

Example: $U(1)^N$ Chern–Simons theory

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Input data: $K \in \text{Mat}_N(\mathbb{Z}) \rightarrow \mathcal{C}$: MTC $\sim$ Cat. with $\otimes$, $\oplus$, semisimp. . . .

Wilson loops: $a, b, \dots \in \mathbb{Z}^N / K\mathbb{Z}^N \rightarrow a, b, \dots \in \text{Simp}(\mathcal{C})$

Fusions: $a + b \in \mathbb{Z}^N / K\mathbb{Z}^N \rightarrow a \times b = \sum_c N_{ab}^c c$

Topological spins: $\theta(a) := \exp(\pi i a^T K^{-1} a) \rightarrow \theta : \mathcal{C} \rightarrow U(1)$

Braidings: $B(a, b) := \exp(2\pi i a^T K^{-1} b) \rightarrow B : \mathcal{C} \times \mathcal{C} \rightarrow U(1)$

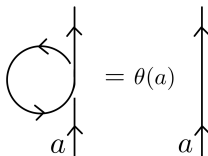
What is TQFT

Anyons: Generalized concepts of bosons and fermions.

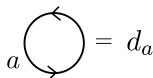
Fusion rule:

$$a \times b = \sum_c N_{ab}^c c, \quad a, b, c \in \{\text{Anyons}\} =: \mathcal{C}, \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}$$

Topological spin: $\theta : \mathcal{C} \rightarrow U(1)$ encoding spin statistics of anyons:


$$\text{Diagram of a line with a loop} = \theta(a)$$

Quantum dimension:


$$\text{Diagram of a loop} = d_a$$

Summary: Modular Tensor Category $\mathcal{C}$ naturally encodes these data.

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What is TQFT with time-reversal symmetry

Recall: $U(1)^N$ Chern–Simons theory

$$S = \frac{1}{4\pi} \int_M K^{IJ} A_I \wedge dA_J$$

With time-reversal, we have

$$\frac{1}{4\pi} \int_M K^{IJ} A_I \wedge dA_J \rightarrow \frac{1}{4\pi} \int_{-M} K^{IJ} A_I \wedge dA_J.$$

Define $\mathsf{T} \in \text{Mat}_N(\mathbb{Z})$ satisfying $\mathsf{T}^T K \mathsf{T} = -K$.

Then we obtain

$$S = \frac{1}{4\pi} \int_{-M} K^{IJ} A_I \wedge dA_J = \frac{1}{4\pi} \int_M K^{IJ} (\mathsf{T}A)_I \wedge d(\mathsf{T}A)_J.$$

What is TQFT with time-reversal symmetry

Example: $U(1)^2$ Chern–Simons theory

$$S = \frac{1}{4\pi} \int_M K^{IJ} A_I \wedge dA_J = \frac{1}{4\pi} \int_M (A_1, A_2) \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} dA_1 \\ dA_2 \end{pmatrix}.$$

Then $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ satisfies $T^T K T = -K$.

$$\begin{aligned} S &= \frac{1}{4\pi} \int_{-M} (A_1, A_2) \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} dA_1 \\ dA_2 \end{pmatrix} \\ &= \frac{1}{4\pi} \int_M (A_1, A_2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} dA_1 \\ dA_2 \end{pmatrix} \\ &= \frac{1}{4\pi} \int_M (A_2, -A_1) \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} dA_2 \\ -dA_1 \end{pmatrix}. \end{aligned}$$

What is TQFT with time-reversal symmetry

Observation: Time-reversal induces $(A_1, A_2) \mapsto (A_2, -A_1)$

$$\mathsf{T} : \exp(\pi i \oint_{\gamma} A_1) \mapsto \exp(\pi i \oint_{\gamma} A_2).$$

Symmetry fractionalization: Categorically, we introduce

$$\mathsf{T} : \mathcal{C} \rightarrow \mathcal{C}, a \mapsto \mathsf{T}a.$$

This operation must respect the pentagon/hexagon conditions.
If this action is well-defined, additional data can be defined:

$$\eta(a) = \begin{cases} \pm 1 & a = \mathsf{T}a, \\ 0 & a \neq \mathsf{T}a. \end{cases}$$

Obstruction: Defining $\eta(a)$ may be obstructed by H^3 -obstruction valued in the twisted group cohomology of degree 3.

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Reshetikhin–Turaev TQFT

Reshetikhin–Turaev (RT) invariant:

$\langle \rangle : M \rightarrow \mathbb{C}$ where M is an **orientable** closed three-manifold.

- We can construct Reshetikhin–Turaev TQFT from this invariant.
- Reshetikhin–Turaev TQFT is understood as a mathematical formulation of Chern–Simons theories.
- This invariant is constructed via Lickorish–Wallece Theorem.

Lickorish–Wallece Theorem:

For any closed connected orientable three-manifold M , there exists a framed link L such that $S_L^3 \cong M$, where S_L^3 is a three-manifold obtained from S^3 via surgery along L .

- Framed link $\sim$ Link with an integer $\in \mathbb{Z}$.
- Surgery $\rightarrow$ See the next slide.

Reshetikhin–Turaev TQFT

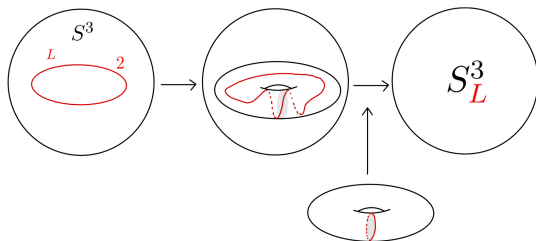


Figure: Rough picture of the Surgery

$$M \cong S_L^3 \rightarrow \langle M \rangle \sim \sum_{a \in \mathcal{C}} \langle L(a) \rangle$$

Here, we use MTC to evaluate knot invariants $\langle L(a.b, \dots) \rangle$.

Summary: $S^3 \xrightarrow{L=\sqcup_i L_i} S_L^3 \cong M \xrightarrow{\mathcal{C}} \langle M \rangle \rightarrow \text{TQFT}$

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Toward formulation of non-orientable TQFT

We have two problems:

- **Algebraically:**
 $\mathcal{C}$ with time-reversal $\mathbb{Z}_2^T \curvearrowright \mathcal{C}$ is not fully understood.
- **Topologically:**
RT invariant can be applied only for orientable manifolds.

However, the situation is not so bad. From a physical perspective, There seems to be a real possibility of constructing TQFTs on non-orientable manifolds.

Toward formulation of non-orientable TQFT

Recall: Algebraically, $N_{ab}^c = \sum_d S_{ad} S_{bd} \overline{S_{cd}} / S_{0d} \in \mathbb{Z}_{\geq 0}$ is nontrivial. From TQFT perspectives, $N_{ab}^c = \dim V(S^2(a, b, \bar{c})) \in \mathbb{Z}_{\geq 0}$ is natural.

Given $\mathcal{C}$ and time-reversal $\mathbb{Z}_2^T \curvearrowright \mathcal{C}$ without H^3 -obstruction:

$$M_a := \sum_b S_{ab} \eta(b)$$

RHS is purely determined with MTC. Surprisingly, we have the followings:

Fact:

- $M_a \in \mathbb{Z}_{\geq 0}$ for every known examples.
 - $M_a \in \mathbb{Z}_{\geq 0}$ is shown for abelian TQFTs. [IO 2502.13532]
 - H^3 -obstruction vanishes for abelian TQFTs. [IO 2509.07368]
- These are highly nontrivial and give some implications.

Insights from crosscap

Recall: By the definition of TQFT $Z : \text{Bord}_{2,3}(\mathcal{C}) \rightarrow \text{Vect}_{\mathbb{C}}$,

- $Z(M) \in V(\partial M)$
- $Z(\Sigma \times S^1) = \dim V(\Sigma)$

where M is a three-manifold, Σ is a closed surface, and $V(\Sigma)$ is a Hilbert space on Σ assigned by TQFT.

→ We can construct bases on $V(T^2)$ as follows:

- **Constructing a basis:**

$$|a\rangle := Z(D^2(a) \times S^1) = Z(\text{disk with red dot} \times S^1) = Z(\text{torus with red dot}) \in V(T^2),$$
$$\rightarrow V(T^2) = \text{span}_{\mathbb{C}} \{|a\rangle \mid a \in \mathcal{C}\}.$$

- **Another basis via modular S :**

$$S|a\rangle := Z(S^1 \times D^2(a)) \in V(T^2),$$
$$\rightarrow V(T^2) = \text{span}_{\mathbb{C}} \{S|a\rangle \mid a \in \mathcal{C}\}.$$

Insights from crosscap

The **Crosscap** state is defined as:

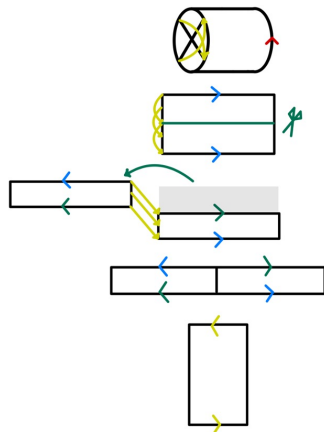
$$\begin{aligned} |\mathbb{C}\mathbb{C}\rangle &= Z(\text{⊗} \times S^1) \\ &= \sum_a M_a |a\rangle \\ &= \sum_a \eta(a) S |a\rangle, \end{aligned}$$

where M_a and $\eta(a)$ are coefficients.

Then, we have

$$M_a = \langle a | \sum_b \eta(b) S |b\rangle = \sum_b S_{ab} \eta(b)$$

$$\eta(a) = \langle a | S^{-1} |\mathbb{C}\mathbb{C}\rangle$$



Remark: $\eta(a)$ is understood as symmetry fractionalization.
[Tachikawa-Yonekura 1611.01601]

Insights from crosscap

We can compute M_a as follows:

$$\begin{aligned} M_a &= \langle a | \mathbb{C}\mathbb{C} \rangle \\ &= Z(\text{disk with red dot} \times S^1) Z(\text{cylinder with red dot} \times S^1) \\ &= Z(\text{cylinder with red dot} \times S^1) \\ &= Z(\text{crosscap with red dot} \times S^1) \\ &= \dim V(\mathbb{R}P^2(a)) \end{aligned}$$

where we used $Z(\Sigma \times S^1) = \dim V(\Sigma)$ for closed surfaces.

If TQFT exists, $M_a \in \mathbb{Z}_{\geq 0}$ has a natural interpretation.

→ Suggestion for the existence of TQFTs.

We can compute $\eta(a)$ as follows: [Tachikawa-Yonekura 1611.01601]

$$\begin{aligned}\eta(a) &= \langle a | S^{-1} | \mathbb{C}\mathbb{C} \rangle \\ &= \dots \\ &= Z \left(\frac{S^2(a, \overline{1a}) \times [0, 1]}{(0, x) \sim (1, -x)} \right) \\ &=: Z(X^3(a))\end{aligned}$$

Recall: $\eta(a)$ is understood as symmetry fractionalization.

But it has another crucial implication.

X^3 is the building block of non-orientable manifolds ! [Lickorish 1963]

Future work:

- $\forall M:\text{orientable}, S^3 \xrightarrow{\exists L} S_L^3 \cong M \xrightarrow{\mathcal{C}} \langle M \rangle \rightarrow \text{TQFT}.$
- $\forall M:\text{non-orientable}, X^3 \xrightarrow{\exists L} X_L^3 \cong M \xrightarrow{\mathbb{Z}_2^T \curvearrowright \mathcal{C}} \langle M \rangle \rightarrow \text{TQFT} ?$

Here, the red part remains to be understood.

We need to know what kind of $\mathcal{C}$ and additional structures are the right data to construct the invariant and TQFTs for non-orientable manifolds.

Physics gives us the clue!

Thank you for listening.